Macroeconomic Theory: A Dynamic General Equilibrium Approach

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Exercises and Solutions

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Exercises

Chapter2

2.1. We have assumed that the economy discounts s periods ahead using the geometric (or exponential) discount factor $\beta^s = (1+\theta)^{-s}$ for $\{s = 0, 1, 2, ...\}$. Suppose instead that the economy uses the sequence of hyperbolic discount factors $\beta_s = \{1, \varphi\beta, \varphi\beta^2, \varphi\beta^3, ...\}$ where $0 < \varphi < 1$.

(a) Compare the implications for discounting of using geometric and hyperbolic discount factors.

(b) For the centrally planned model

$$\begin{aligned} y_t &= c_t + i_t \\ \Delta k_{t+1} &= i_t - \delta k_t \end{aligned}$$

where y_t is output, c_t is consumption, i_t is investment, k_t is the capital stock and the objective is to maximize

$$V_t = \sum_{s=0}^{\infty} \beta_s U(c_{t+s})$$

derive the optimal solution under hyperbolic discounting and comment on any differences with the solution based on geometric discounting.

2.2. Assuming hyperbolic discounting, the utility function $U(c_t) = \ln c_t$ and the production function $y_t = Ak_t$,

(a) derive the optimal long-run solution.

(b) Analyse the short-run solution.

2.3. Consider the CES production function $y_t = A[\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha)n_t^{1-\frac{1}{\gamma}}]^{\frac{1}{1-\frac{1}{\gamma}}}$

(a) Show that the CES function becomes the Cobb-Douglas function as $\gamma \to 1$.

(b) Verify that the CES function is homogeneous of degree one and hence satisfies $F(k_t, n_t) = F_{n,t}n_t + F_{k,t}k_t$.

2.4. Consider the following centrally-planned model with labor

$$\begin{aligned}
y_t &= c_t + i_t \\
\Delta k_{t+1} &= i_t - \delta k_t \\
y_t &= A[\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}]^{\frac{1}{1-\frac{1}{\gamma}}}
\end{aligned}$$

where the objective is to maximize

$$V_t = \sum_{s=0}^{\infty} \beta^s [\ln c_{t+s} + \varphi \ln l_{t+s}], \quad \beta = \frac{1}{1+\theta}$$

where y_t is output, c_t is consumption, i_t is investment, k_t is the capital stock, n_t is employment and l_t is leisure ($l_t + n_t = 1$).

(a) Derive expressions from which the long-run solutions for consumption, labour and capital may be obtained.

(b) What are the implied long-run real interest rate and wage rate?

(c) Comment on the implications for labor of having an elasticity of substitution between capital and labor different from unity

(d) Obtain the long-run capital-labor ratio.

2.5. (a) Comment on the statement: "the saddlepath is a knife-edge solution; once the economy departs from the saddlepath it is unable to return to equilibrium and will instead either explode or collapse."

(b) Show that although the solution for the basic centrally-planned economy of Chapter 2 is a saddlepath, it can be approximately represented by a stable autoregressive process.

2.6. In continuous time the basic centrally-planned economy problem can be written as: maximize $\int_0^{\infty} e^{-\theta t} u(c_t) dt$ with respect $\{c_t,\ k_t\}$ subject to the budget constraint $F(k_t) = c_t + \dot{k}_t + \delta k_t$.

(a) Obtain the solution using the Calculus of Variations.

(b) Obtain the solution using the Maximum Principle.

(c) Compare these solutions with the discrete-time solution of Chapter 2.

Chapter 3

3.1. Re-work the optimal growth solution in terms of the original variables, i.e. without first taking deviations about trend growth.

(a) Derive the Euler equation

(b) Discuss the steady-state optimal growth paths for consumption, capital and output.

3.2. Consider the Solow-Swan model of growth for the constant returns to scale production function $Y_t = F[e^{\mu t}K_t, e^{\nu t}N_t]$ where μ and ν are the rates of capital and labor augmenting technical progress.

(a) Show that the model has constant steady-state growth when technical progress is labor augmenting.

(b) What is the effect of the presence of non-labor augmenting technical progress?

3.3. Consider the Solow-Swan model of growth for the production function $Y_t = A(e^{\mu t}K_t)^{\alpha}(e^{\nu t}N_t)^{\beta}$ where μ is the rate of capital augmenting technical progress and ν is the rate of labor augmenting technical progress. Consider whether a steady-state growth solution exists under

(a) increasing returns to scale, and

(b) constant returns to scale.

(c) Hence comment on the effect of the degree of returns to scale on the rate of economic growth, and the necessity of having either capital or labor augmenting technical progress in order to achieve economic growth.

3.4. Consider the following two-sector endogenous growth model of the economy due to Rebelo (1991) which has two types of capital, physical k_t and human h_t. Both types of capital are required to produce goods output y_t and new human capital i_t^h. The model is

$$\begin{aligned} y_t &= c_t + i_t^k \\ \Delta k_{t+1} &= i_t^k - \delta k_t \\ \Delta h_{t+1} &= i_t^h - \delta h_t \\ y_t &= A(\phi k_t)^\alpha (\mu h_t)^{1-\alpha} \\ i_t^h &= A[(1-\phi)k_t]^\varepsilon [(1-\mu)h_t]^{1-\varepsilon} \end{aligned}$$

where i_t^k is investment in physical capital, ϕ and μ are the shares of physical and human capital used in producing goods and $\alpha > \varepsilon$. The economy maximizes $V_t = \Sigma_{s=0}^{\infty} \beta^s \frac{c_{t+s}^{1-\sigma}}{1-\sigma}$.

(a) Assuming that each type of capital receives the same rate of return in both activities, find the steady-state ratio of the two capital stocks

(b) Derive the optimal steady-state rate of growth.

(c) Examine the special case of $\varepsilon = 0$.

Chapter 4

4.1. The household budget constraint may be expressed in different ways from equation (4.2) where the increase in assets from the start of the current to the next period equals total income less consumption. Derive the Euler equation for consumption and compare this with the solution based on equation (4.2) for each of the following ways of writing the budget constraint:

(a) $a_{t+1} = (1+r)(a_t + x_t - c_t)$, i.e. current assets and income assets that are not consumed are invested.

(b) $\Delta a_t + c_t = x_t + r a_{t-1}$, where the dating convention is that a_t denotes the end of period stock of assets and c_t and x_t are consumption and income during period t.

(c) $W_t = \Sigma_{s=0}^{\infty} \frac{c_{t+s}}{(1+r)^s} = \Sigma_{s=0}^{\infty} \frac{x_{t+s}}{(1+r)^s} + (1+r)a_t$, where W_t is household wealth.

4.2. The representative household is assumed to choose $\{c_t, c_{t+1}, ...\}$ to maximise $V_t = \sum_{s=0}^{\infty} \beta^s U(c_{t+s})$, $0 < \beta = \frac{1}{1+\theta} < 1$ subject to the budget constraint $\Delta a_{t+1} + c_t = x_t + r_t a_t$ where c_t is consumption,

x_t is exogenous income, a_t is the (net) stock of financial assets at the beginning of period t and r is the (constant) real interest rate.

(a) Assuming that $r = \theta$ and using the approximation $\frac{U'(c_{t+1})}{U'(c_t)} = 1 - \sigma\Delta \ln c_{t+1}$, show that optimal consumption is constant.

(b) Does this mean that changes in income will have no effect on consumption? Explain.

4.3 (a) Derive the dynamic path of optimal household consumption when the utility function reflects exogenous habit persistence h_t and the utility function is $U(c_t) = \frac{(c_t - h_t)^{1-\sigma}}{1-\sigma}$. and household budget constraint is $\Delta a_{t+1} + c_t = x_t + ra_t$.

(b) Hence, obtain the consumption function making the assumption that $\beta(1 + r) = 1$. Comment on the case where expected future levels of habit persistence are the same as those in the current period, i.e. $h_{t+s} = h_t$ for $s \geq 0$.

4.4. Derive the behavior of optimal household consumption when the utility function reflects habit persistence of the following forms:

(a) $U(c_t) = -\frac{(c_t - \gamma c_{t-1})^2}{2} + \alpha(c_t - \gamma c_{t-1})$

(b) $U(c_t) = \frac{(c_t - \gamma c_{t-1})^{1-\sigma}}{1-\sigma}$.

where the budget constraint is $\Delta a_{t+1} + c_t = x_t + ra_t$.

(c) Compare (b) with the case where $U(c_t) = \frac{(c_t - h_t)^{1-\sigma}}{1-\sigma}$ and h_t is an exogenous habitual level of consumption.

4.5. Suppose that households have savings of s_t at the start of the period, consume c_t but have no income. The household budget constraint is $\Delta s_{t+1} = r(s_t - c_t)$, $0 < r < 1$ where r is the real interest rate.

(a) If the household's problem is to maximize discounted utility $V_t = \Sigma_{s=0}^{\infty} \beta^s \ln c_{t+s}$where $\beta = \frac{1}{1+r}$,

(i) show that the solution is $c_{t+1} = c_t$

(ii) What is the solution for s_t?

(b) If the household's problem is to maximize expected discounted utility $V_t = E_t \Sigma_{s=0}^{\infty} \beta^s \ln c_{t+s}$

(i) show that the solution is $\frac{1}{c_t} = E_t[\frac{1}{c_{t+1}}]$

(ii) Using a second-order Taylor series expansion about c_t show that the solution can be written as $E_t[\frac{\Delta c_{t+1}}{c_t}] = E_t[(\frac{\Delta c_{t+1}}{c_t})^2]$

(iii) Hence, comment on the differences between the non-stochastic and the stochastic solutions.

4.5. Suppose that households seek to maximize the inter-temporal quadratic objective function

$$V_t = -\frac{1}{2} E_t \sum_{s=0}^{\infty} \beta^s [(c_{t+s} - \gamma)^2 + \phi(a_{t+s+1} - a_{t+s})^2], \qquad \beta = \frac{1}{1+r}$$

subject to the budget constraint

$$c_t + a_{t+1} = (1+r)a_t + x_t$$

where c_t is consumption, a_t is the stock of assets and x_t is exogenous.

(a) Comment on the objective function.

(b) Derive expressions for the optimal dynamic behaviors of consumption and the asset stock.

(c) What is the effect on consumption and assets of a permanent shock to x_t of Δx? Comment on the implications for the specification of the utility function.

(d) What is the effect on consumption and assets of a temporary shock to x_t that is unanticipated prior to period t?

(e) What is the effect on consumption and assets of a temporary shock to x_{t+1} that is anticipated in period t?

4.6. Households live for periods t and $t+1$. The discount factor for period $t+1$ is $\beta = 1$. They receive exogenous income x_t and x_{t+1}, where the conditional distribution of income in period $t+1$ is $N(x_t, \sigma^2)$, but they have no assets.

(a) Find the level of c_t that maximises $V_t = U(c_t) + E_t U(c_{t+1})$ if the utility function is quadratic: $U(c_t) = -\frac{1}{2}c_t^2 + \alpha c_t$, $(\alpha > 0)$.

(b) Calculate the conditional variance of this level of c_t and hence comment on what this implies about consumption smoothing.

4.7. An alternative way of treating uncertainty is through the use of contingent states s^t, which denotes the state of the economy up to and including time t, where $s^t = (s_t, s_{t-1},...)$ and there are S different possible states with probabilities $p(s^t)$. The aim of the household can then be expressed as maximizing over time and over all possible states of nature the expected discounted sum of current and future utility

$$\Sigma_{t,s}\beta^t p(s^t)U[c(s^t)]$$

subject to the budget constraint in state s^t

$$c(s^t) + a(s^t) = [1 + r(s^t)]a(s^{t-1}) + x(s^t)$$

where $c(s^t)$ is consumption, $a(s^t)$ are assets and $x(s^t)$ is exogenous income in state s^t. Derive the optimal solutions for consumption and assets.

4.8. Suppose that firms face additional (quadratic) costs associated with the accumulation of capital and labor so that firm profits are

$$\Pi_t = Ak_t^\alpha n_t^{1-\alpha} - w_t n_t - i_t - \frac{1}{2}\mu(\Delta k_{t+1})^2 - \frac{1}{2}\nu(\Delta n_{t+1})^2$$

where $\mu, \nu > 0$, the real wage w_t is exogenous and $\Delta k_{t+1} = i_t - \delta k_t$. If firms maximize the expected present value of the firm $E_t[\Sigma_{s=0}^{\infty}(1+r)^{-s}\Pi_{t+s}]$,

(a) derive the demand functions for capital and labor in the long run and the short run.

(b) What would be the response of capital and labor demand to

(i) a temporary increase in the real wage in period t, and

(ii) a permanent increase in the real wage from period t?

Chapter 5

5.1. In an economy that seeks to maximize $\Sigma_{s=0}^{\infty}\beta^s \ln c_{t+s}$ ($\beta = \frac{1}{1+r}$) and takes output as given the government finances its expenditures by taxes on consumption at the rate τ_t and by debt.

(a) Find the optimal solution for consumption given government expendtitures, tax rates and government debt.

(b) Starting from a position where the budget is balanced and there is no government debt, analyse the consequences of

(i) a temporary increase in government expenditures in period t,

(ii) a permanent increase in government expenditures from period t.

5.2. Suppose that government expenditures g_t are all capital expenditures and the stock of government capital G_t is a factor of production. If the economy is described by

$$\begin{aligned} y_t &= c_t + i_t + g_t \\ y_t &= Ak_t^{\alpha} G_t^{1-a} \\ \Delta k_{t+1} &= i_t - \delta k_t \\ \Delta G_{t+1} &= g_t - \delta G_t \end{aligned}$$

and the aim is to maximize $\Sigma_{s=0}^{\infty}\beta^s \ln c_{t+s}$,

(a) obtain the optimal solution.

(b) Comment on how government expenditures are being implicitly paid for in this problem.

5.3. Suppose that government finances its expenditures through lump-sum taxes T_t and debt b_t but there is a cost of collecting taxes given by

$$\Phi(T_t) = \phi_1 T_t + \frac{1}{2}\phi_2 T_t^2, \qquad \Phi'(T_t) \geq 0$$

If the national income identity and the government budget constraint are

$$\begin{aligned} y_t &= c_t + g_t + \Phi(T_t) \\ \Delta b_{t+1} + T_t &= g_t + r b_t + \Phi(T_t) \end{aligned}$$

where output y_t and government expenditures g_t are exogenous, and the aim is to maximize $\Sigma_{s=0}^{\infty}\beta^s U(c_{t+s})$ for $\beta = \frac{1}{1+r}$,

(a) find the optimal solution for taxes.

(b) What is the household budget constraint?

(c) Analyse the effects on taxes, debt and consumption of

(i) a temporary increase government expenditures in period t

(ii) an increase in output.

5.4. Assuming that output growth is zero, inflation and the rate of growth of the money supply are π, that government expenditures on goods and services plus transfers less total taxes equals z and the real interest rate is $r > 0$,

(a) what is the minimum rate of inflation consistent with the sustainability of the fiscal stance in an economy that has government debt?

(b) How do larger government expenditures affect this?

(c) What are the implications for reducing inflation?

5.5. Consider an economy without capital that has proportional taxes on consumption and labor and is described by the following equations

$$
\begin{aligned}
y_t &= An_t^{\alpha} = c_t + g_t \\
g_t + rb_t &= \tau_t^c c_t + \tau_t^w w_t n_t + \Delta b_{t+1} \\
U(c_t, l_t) &= \ln c_t + \gamma \ln l_t \\
1 &= n_t + l_t
\end{aligned}
$$

(a) State the household budget constraint.

(b) If the economy seeks to maximize $\Sigma_{s=0}^{\infty}\beta^s U(c_{t+s}, l_{t+s})$, where $\beta = \frac{1}{1+r}$, derive the optimal steady-state levels of consumption and employment for given g_t, b_t and tax rates.

5.6 (a) What is the Ramsey problem of optimal taxation?

(b) For Exercise 5 find the optimal rates of consumption and labor taxes by solving the associated Ramsey problem.

Chapter 6

6.1. (a) Consider the following two-period OLG model. People consume in both periods but work only in period two. The inter-temporal utility of the representative individual in the first period is

$$\mathcal{U} = \ln c_1 + \beta[\ln c_2 + \alpha \ln(1 - n_2) + \gamma \ln g_2]$$

where c_1 and c_2 are consumption and k_1 (which is given) and k_2 are the stocks of capital in periods one and two, n_2 is work and g_2 is government expenditure in period two which is funded by a lump-sum tax in period two. Production in periods one and two are

$$\begin{aligned} y_1 &= Rk_1 = c_1 + k_2 \\ y_2 &= Rk_2 + \phi n_2 = c_2 + g_2 \end{aligned}$$

Find the optimal centrally-planned solution for c_1.

(b) Find the private sector solutions for c_1 and c_2, taking government expenditures as given.

(c) Compare the two solutions.

6.2 Suppose that in Exercise 6.1 the government finances its expenditures with taxes both on labor and capital in period two so that the government budget constraint is

$$g_2 = \tau_2 \phi n_2 + (R - R_2)k_2$$

where R_2 is the after-tax return to capital and τ_2 is the rate of tax of labor in period two. Derive the centrally-planned solutions for c_1 and c_2.

6.3. (a) Continuing to assume that the government budget constraint is as defined in Exercise 6.2, find the private sector solutions for c_1 and c_2 when government expenditures and tax rates are pre-announced.

(b) Why may this solution not be time consistent?

6.4 For Exercise 6.3 assume now that the government optimizes taxes in period two taking k_2 as given as it was determined in period one.

(a) Derive the necessary conditions for the optimal solution.

(b) Show that the optimal labor tax when period two arrives is zero. Is it optimal to taxe capital in period two?

6.5. Consider the following two-period OLG model in which each generation has the same number of people, N. The young generation receives an endowment of x_1 when young and $x_2 = (1 + \phi)x_1$ when old, where ϕ can be positive or negative. The endowments of the young generation grow over time at the rate γ. Each unit of saving (by the young) is invested and produces $1+\mu$ units of output ($\mu > 0$) when they are old. Each of the young generation maximizes $\ln c_{1t} + \frac{1}{1+r} \ln c_{2,t+1}$, where c_{1t} is consumption when young and $c_{2,t+1}$ is consumption when old.

(a) Derive the consumption and savings of the young generation and the consumption of the old generation.

(b) How do changes in ϕ, μ, r and γ affect these solutions?

(c) If $\phi = \mu$ how does this affect the solution?

Chapter 7

7.1. An open economy has the balance of payments identity

$$x_t - Qx_t^m + r^* f_t = \Delta f_{t+1}$$

where x_t is exports, x_t^m is imports, f_t is the net holding of foreign assets, Q is the terms of trade and r^* is the world rate of interest. Total output y_t is either consumed at home c_t^h or is exported, thus

$$y_t = c_t^h + x_t.$$

Total domestic consumption is c_t; y_t and x_t are exogenous.

(a) Derive the Euler equation that maximises $\sum_{s=0}^{\infty} \beta^s \ln c_{t+s}$ with respect to $\{c_t, c_{t+1},...; f_{t+1}, f_{t+2},...\}$ where $\beta = \frac{1}{1+\theta}$.

(b) Explain how and why the relative magnitudes of r^* and θ affect the steady-state solutions of c_t and f_t.

(c) Explain how this solution differs from that of the corresponding closed-economy.

(d) Comment on whether there are any benefits to being an open economy in this model.

(e) Obtain the solution for the current account.

(f) What are the effects on the current account and the net asset position of a permanent increase in x_t?

7.2. Consider two countries which consume home and foreign goods $c_{H,t}$ and $c_{F,t}$. Each period the home country maximizes

$$U_t = \left[c_{H,t}^{\frac{\sigma-1}{\sigma}} + c_{F,t}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

and has an endowment of y_t units of the home produced good. The foreign country is identical and its variables are denoted with an asterisk. Every unit of a good that is transported abroad has a real resource cost equal to τ so that, in effect, only a proportion $1 - \tau$ arrives at its destination. $P_{H,t}$ is the home price of the home good and $P_{H,t}^*$ is the foreign price of the home good. The corresponding prices of the foreign good are $P_{F,t}$ and $P^*_{F,t}$. All prices are measured in terms of a common unit of world currency.

(a) If goods markets are competitive what is the relation between the four prices and how are the terms of trade in each country related?

(b) Derive the relative demands for home and foreign goods in each country.

(c) Hence comment on the implications of the presence of transport costs.

Note: This Exercise and the next, Exercise 7.3, is based on Obstfeld and Rogoff (2000).

7.3. Suppose the model in Exercise 7.2 is modified so that there are two periods and intertemporal utility is

$$V_t = U(c_t) + \beta U(c_{t+1})$$

where $c_t = \left[c_{H,t}^{\frac{\sigma-1}{\sigma}} + c_{F,t}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$. Endowments in the two periods are y_t and y_{t+1}. Foreign prices $P^*_{H,t}$ and $P^*_{F,t}$ and the world interest rate are assumed given. The first and second period budget constraints are

$$\begin{aligned} P_{H,t}y_t + B &= P_{H,t}c_{H,t} + P_{F,t}c_{F,t} = P_t c_t \\ P_{H,t+1}y_{t+1} - (1+r^*)B &= P_{H,t+1}c_{H,t+1} + P_{F,t+1}c_{F,t+1} = P_{t+1}c_{t+1}, \end{aligned}$$

where P_t is the general price level, B is borrowing from abroad in world currency units in period t and r^* is the foreign real interest rate. It is assumed that there is zero foreign inflation.

(a) Derive the optimal solution for the home economy, including the domestic price level P_t.

(b) What is the domestic real interest rate r? Does real interest parity exist?

(c) How is r related to τ?

7.4. Suppose the "world" is compromised of two similar countries where one is a net debtor. Each country consumes home and foreign goods and maximizes

$$V_t = \sum_{s=0}^{\infty} \beta^s \frac{(c_{H,t+s}^{\alpha} c_{F,t+s}^{1-\alpha})^{1-\sigma}}{1-\sigma}$$

subject to its budget constraint. Expressed in terms of home's prices, the home country budget constraint is

$$P_{H,t}c_{H,t} + S_t P_{F,t}c_{F,t} + \Delta B_{t+1} = P_{H,t}y_{H,t} + R_t B_t$$

where $c_{H,t}$ is consumption of home produced goods, $c_{F,t}$ is consumption of foreign produced goods, $P_{H,t}$ is the price of the home country's output which is denoted $y_{H,t}$ and is exogenous, $P_{F,t}$ is the price of the foreign country's output in terms of foreign prices, and B_t is the home country's borrowing from abroad expressed in domestic currency which is at the nominal rate of interest R_t and S_t is the nominal exchange rate. Interest parity is assumed to hold.

(a) Using an asterisk to denote the foreign country equivalent variable (e.g. $c^*_{H,t}$ is the foreign country's consumption of domestic output), what are the national income and balance of payments identities for the home country?

(b) Derive the optimal relative expenditure on home and foreign goods taking the foreign country - its output, exports and prices - and the exchange rate as given.

(c) Derive the price level P_t for the domestic economy assuming that $c_t = c^{\alpha}_{H,t+s}c^{1-\alpha}_{F,t+s}$.

(d) Obtain the consumption Euler equation for the home country.

(e) Hence derive the implications for the current account and the net foreign asset position. Comment on the implications of the home country being a debtor nation.

(f) Suppose that $y_t < y^*_t$ and both are constant, that there is zero inflation in each country, $R_t = R$ and $\beta = \frac{1}{1+R}$. Show that $c_t < c^*_t$ if $B_t \geq 0$.

7.5. For the model described in Exercise 7.4, suppose that there is world central planner who maximizes the sum of individual country welfares:

$$W_t = \sum_{s=0}^{\infty} \beta^s \left[\frac{(c^{\alpha}_{H,t+s}c^{1-\alpha}_{F,t+s})^{1-\sigma}}{1-\sigma} + \frac{[(c^*_{H,t+s})^{\alpha}(c^*_{F,t+s})^{1-\alpha}]^{1-\sigma}}{1-\sigma} \right].$$

(a) What are the constraints in this problem?

(b) Derive the optimal world solution subject to these constraints where outputs and the exchange rate are exogenous.

(c) Comment on any differences with the solutions in Exercise 7.4.

Chapter 8

8.1. Consider an economy in which money is the only financial asset, and suppose that households hold money solely in order to smooth consumption expenditures. The nominal household budget constraint for this economy is

$$P_t c_t + \Delta M_{t+1} = P_t y_t$$

where c_t is consumption, y_t is exogenous income, P_t is the price level and M_t is nominal money balances.

(a) If households maximize $\Sigma_{s=0}^{\infty}\beta^s U(c_{t+s})$, derive the optimal solution for consumption.

(b) Compare this solution with the special case where $\beta = 1$ and inflation is zero.

(c) Suppose that in (b) y_t is expected to remain constant except in period $t+1$ when it is expected to increase temporarily. Examine the effect on money holdings and consumption.

(d) Hence comment on the role of real balances in determining consumption in these circumstances.

8.2. Suppose that the nominal household budget constraint is

$$\Delta B_{t+1} + \Delta M_{t+1} + P_t c_t = P_t x_t + R_t B_t$$

where c_t is consumption, x_t is exogenous income, B_t is nominal bond holding, M_t is nominal money balances, P_t is the general price level, $m_t = M_t/P_t$ and R_t is a nominal rate of return.

(a) Derive the real budget constraint.

(b) Comment on whether or not this implies that money is super-neutral in the whole economy.

(c) If households maximize

$$V_t = \Sigma_{s=0}^{\infty}\beta^s U(c_{t+s}, m_{t+s})$$

where the utility function is

$$U(c_t, m_t) = \frac{\left[\frac{c_t^{\alpha} m_t^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\right]^{1-\sigma}}{1-\sigma}$$

obtain the demand for money.

8.3. Consider a cash-in-advance economy with the national income identity

$$y_t = c_t + g_t$$

and the government budget constraint

$$\Delta B_{t+1} + \Delta M_{t+1} + P_t T_t = P_t g_t + R_t B_t$$

where c_t is consumption, y_t is exogenous national income, B_t is nominal bond holding, M_t is nominal money balances, P_t is the general price level, $m_t = M_t/P_t$, T_t are lump-sum taxes, R_t is a nominal rate of return and the government consumes a random real amount $g_t = g + e_t$ where e_t is an independently and identically distributed random shock with zero mean.

(a) If households maximize $\Sigma_{s=0}^{\infty}\beta^s \ln c_{t+s}$ where $\beta = \frac{1}{1+\theta}$, derive the optimal solutions for consumption and money holding.

(b) Comment on how a positive government expenditure shock affects consumption and money holding.

(c) Is money super-neutral in this economy?

8.4. Suppose that some goods $c_{1,t}$ must be paid for only with money M_t and the rest $c_{2,t}$ are bought on credit L_t using a one period loan to be repaid at the start of next period at the nominal rate of interest $R + \rho$, where R is the rate of interest on bonds which are a savings vehicle. The prices of these goods are P_{1t} and P_{2t}. If households maximize $\Sigma_{s=0}^{\infty}(1 + R)^{-s}U(c_{t+s})$ subject to their budget constraint, where $U(c_t) = \ln c_t$, $c_t = \frac{c_{1,t}^{\alpha} c_{2,t}^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}$, and income y_t is exogenous,

(a) derive the expenditures on cash purchases relative to credit.

(b) Obtain the optimal long-run solutions for $c_{1,t}$ and $c_{2,t}$ when exogenous income y_t is constant.

(c) Comment on the case where there is no credit premium.

8.5. Suppose that an economy can either use cash-in-advance or credit. Compare the long-run levels of consumption that result from these choices for the economy in Exercise 8.4 when there is a single consumption good c_t.

8.6. Consider the following demand for money function which has been used to study hyper-inflation

$$m_t - p_t = -\alpha(E_t p_{t+1} - p_t), \;\; \alpha > 0$$

where M_t= nominal money, $m_t = \ln M_t$, P_t= price level and $p_t = \ln P_t$.

(a) Contrast this with a more conventional demand function for money, and comment on why it might be a suitable formulation for studying hyper-inflation?

(b) Derive the equilibrium values of p_t and the rate of inflation if the supply of money is given by

$$\Delta m_t = \mu + \varepsilon_t$$

where $\mu > 0$ and $E_t[\varepsilon_{t+1}] = 0$.

(c) What will be the equilibrium values of p_t if

(i) the stock of money is expected to deviate temporarily in period $t+1$ from this money supply rule and take the value m^*_{t+1},

(ii) the rate of growth of money is expected to deviate permanently from the rule and from period $t+1$ grow at the rate υ.

Chapter 9

9.1. Consider an economy that produces a single good in which households maximize

$$V_t = \sum_{s=0}^{\infty} \beta^s \left[\ln c_{t+s} - \phi \ln n_{t+s} + \gamma \ln \frac{M_{t+s}}{P_{t+s}} \right], \qquad \beta = \frac{1}{1+r}$$

subject to the nominal budget constraint

$$P_t c_t + \Delta B_{t+1} + \Delta M_{t+1} = P_t d_t + W_t n_t + R B_t.$$

where c consumption, n is employment, W is the nominal wage rate, d is total real firm net revenues distributed as dividends, B is nominal bond holdings, R is the nominal interest rate, M is nominal money balances, P is the price level and r is the real interest rate. Firms maximize the present value of nominal net revenues

$$\Pi_t = \sum_{s=0}^{\infty} (1+r)^{-s} P_{t+s} d_{t+s}$$

where $d_t = y_t - w_t n_t$, the real wage is $w_t = W_t/P_t$ and the production function is $y_t = A_t n_t^{\alpha}$.

(a) Derive the optimal solution on the assumption that prices are perfectly flexible.

(b) Assuming that inflation is zero, suppose that, following a shock, for example, to the money supply, firms are able to adjust their price with probability ρ, and otherwise price retains its previous value. Discuss the consequences for the expected price level following the shock.

(c) Suppose that prices are fully flexible but the nominal wage adjusts to shocks with probability ρ. What are the consequences for the economy?

9.2. Consider an economy where prices are determined in each period under imperfect competition in which households have the utility function

$$U[c_t, n_t(i)] = \ln c_t - \eta \ln n_t(i)$$

with $i = 1, 2$. Total household consumption c_t is obtained from the two consumption goods $c_t(1)$ and $c_t(2)$ through the aggregator

$$c_t = \frac{c_t(1)^{\phi} c_t(2)^{1-\phi}}{\phi^{\phi}(1-\phi)^{1-\phi}}$$

and $n_t(1)$ and $n_t(2)$ are the employment levels in the two firms which have production functions

$$y_t(i) = A_{it}n_t(i)$$

and profits

$$\Pi_t(i) = P_t(i)y_t(i) - W_t(i)n_t(i)$$

where $P_t(i)$ is the output price and $W_t(i)$ is the wage rate paid by firm i. If total consumption expenditure is

$$P_tc_t = P_t(1)c_t(1) + P_t(2)c_t(2)$$

(a) Derive the optimal solutions for the household, treating firm profits as exogenous.

(b) Show how the price level for each firm is related to the common wage W_t and comment on your result.

9.3. Consider a model with two intermediate goods where final output is related to intermediate inputs through

$$y_t = \frac{y_t(1)^{\phi}y_t(2)^{1-\phi}}{\phi^{\phi}(1-\phi)^{1-\phi}}$$

and the final output producer chooses the inputs $y_t(1)$ and $y_t(2)$ to maximize the profits of the final producer

$$\Pi_t = P_ty_t - P_t(1)y_t(1) - P_t(2)y_t(2)$$

where P_t is the price of final output and $P_t(i)$ are the prices of the intermediate inputs. Intermediate goods are produced with the production function

$$y_t(i) = A_{it}n_t(i)^{\alpha}$$

where $n_t(i)$ is labour input and the intermediate goods firms maximize the profit function

$$\Pi_t(i) = P_t(i)y_t(i) - W_tn_t(i)$$

where W_t is the economy-wide wage rate.

(a) Derive the demand functions for the intermediate inputs.

(b) Derive their supply functions.

(c) Hence examine whether there is an efficiency loss for total output.

9.4. Consider pricing with intermediate inputs where the demand for an intermediate firm's output is

$$y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\phi} y_t$$

its profit is

$$\Pi_t(i) = P_t(i)y_t(i) - C_t(i)$$

and its total cost is

$$C_t(i) = \frac{\phi}{1-\phi}\ln[P_t(i)y_t(i)].$$

(a) Find the optimal price $P_t(i)^*$ if the firm maximizes profits period by period while taking y_t and P_t as given.

(b) If instead the firm chooses a price which it plans to keep constant for all future periods and hence maximizes $\Sigma_{s=0}^{\infty}(1+r)^{-s}\Pi_{t+s}(i)$, derive the resulting optimal price $P_t(i)^{\#}$.

(c) What is this price if expressed in terms of $P_t(i)^*$?

(d) Hence comment on the effect on today's price of anticipated future shocks to demand and costs.

9.5. Consider an economy with two sectors $i = 1, 2$. Each sector sets its price for two periods but does so in alternate periods. The general price level in the economy is the average of sector prices: $p_t = \frac{1}{2}(p_{1t} + p_{2t})$, hence $p_t = \frac{1}{2}(p_{it}^{\#} + p_{i+1,t-1}^{\#})$. In the period the price is reset it is determined by the average of the current and the expected future optimal price: $p_{it}^{\#} = \frac{1}{2}(p_{it}^* + E_t p_{i,t+1}^*)$, $i = 1, 2$. The optimal price is assumed to be determined by $p_{it}^* - p_t = \phi(w_t - p_t)$, where w_t is the wage rate.

(a) Derive the general price level if wages are generated by $\Delta w_t = e_t$, where e_t is a zero mean *i.i.d.* process. Show that p_t can be given a forward-looking, a backward-looking and a univariate representation.

(b) If the price level in steady state is p, how does the price level in period t respond to an unanticipated shock in wages in period t?

(c) How does the price level deviate from p in period t in response to an anticipated wage shock in period $t+1$?

Chapter 10

10.1. (a) Suppose that a consumer's initial wealth is given by W_0, and the consumer has the option of investing in a risky asset which has a rate of return r or a risk-free asset which has a sure rate of return f. If the consumer maximizes the expected value of a strictly increasing, concave utility function $U(W)$ by choosing to hold the risky versus the risk-free asset, and if the variance of the return on the risky asset is $V(r)$, find an expression for the risk premium ρ that makes the consumer indifferent between holding the risky and the risk-free asset.

(b) Explain how absolute risk aversion differs from relative risk aversion.

(c) Suppose that the consumer's utility function is the hyperbolic absolute risk aversion (HARA) function

$$U(W) = \frac{1-\sigma}{\sigma}\left[\frac{\alpha W}{1-\sigma} + \beta\right]^{\sigma}, \quad \alpha > 0,\ \beta > 0, ; 0 < \sigma < 1.$$

Discuss how the magnitude of the risk premium varies as a function of wealth and the parameters α, β, and σ.

10.2. Consider there exists a representative risk-averse investor who derives utility from current and future consumption according to

$$\mathcal{U} = \Sigma_{s=0}^{\theta}\beta^s E_t U(c_{t+s}),$$

where $0 < \beta < 1$ is the consumer's subjective discount factor, and the single-period utility function has the form

$$U(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}, \quad \sigma \geq 0.$$

The investor receives a random exogenous income of y_t and can save by purchasing shares in a stock or by holding a risk-free one-period bond with a face-value of unity. The ex-dividend price of the stock is given by P_t^S in period t. The stock pays a random stream of dividends D_{t+s} per share held at the end of the previous period. The bond sells for P_t^B in period t.

(a) Find an expression for the bond price that must hold at the investor's optimum.

(b) Find an expression for the stock price that must hold at the investor's optimum. Interpret this expression.

(c) Derive an expression for the risk premium on the stock that must hold at the investor's optimum. Interpret this expression.

10.3. If the pricing kernel is M_{t+1}, the return on a risky asset is r_t and that on a risk-free asset is f_t,

(a) state the asset-pricing equation for the risky asset and the associated risk premium.

(b) Express the risk premium as a function of the conditional variance of the risky asset and give a regression interpretation of your result.

10.4. (a) What is the significance of an asset having the same pay-off in all states of the world?

(b) Consider a situation involving three assets and two states. Suppose that one asset is a risk-free bond with a return of 20%, a second asset has a price of 100 and pay-offs of 60 and 200 in the two states, and a third asset has pay-offs of 100 and 0 in the two states. If the probability of the first state occurring is 0.4.

(i) What types of assets might this description fit?

(ii) Find the prices of the implied contingent claims in the two states.

(iii) Find the price of the third asset.

(iv) What is the risk premium associated with the second asset?

10.5. Consider the following two-period problem for a household in which there is one state of the world in the first period and two states in the second period. Income in the first period is 6; in the second period it is 5 in state one which occurs with probability 0.2, and is 10 in state two. There is a risk-free bond with a rate of return equal to 0.2. If instantaneous utility is $\ln c_t$ and the rate of time discount is 0.2 find

(a) the levels of consumption in each state,

(b) the state prices,

(c) the stochastic discount factors,

(d) the risk-free "rate of return" (i.e. rate of change) to income in period two,

(e) the "risk premium" for income in period two.

Chapter 11

11.1. An investor with the utility function $U(c_t) = \frac{c_t^{`-\sigma}}{1-\sigma}$ who maximizes $E_t\Sigma_{s=0}^{\infty}\beta^s U(c_{t+s})$ can either invest in equity with a price of P_t^S and a dividend of D_t or a risk-free one-period bond with nominal return f_t. Derive

(a) the optimal consumption plan, and

(b) the equity premium.

(c) Discuss the effect on the price of equity in period t of a loosening of monetary policy as implemented by an increase in the nominal risk-free rate f_t.

11.2. (a) A household with the utility function $U(c_t) = \ln c_t$, which maximizes $E_t\Sigma_{s=0}^{\infty}\beta^s U(c_{t+s})$, can either invest in a one-period domestic risk-free bond with nominal return R_t, or a one-period foreign currency bond with nominal return (in foreign currency) of R_t^*. If the nominal exchange rate (the domestic price of foreign exchange) is S_t derive

(i) the optimal consumption plan, and

(ii) the foreign exchange risk premium.

(b) Suppose that foreign households have an identical utility function but a different discount factor β^*, what is their consumption plan and their risk premium?

(c) Is the market complete? If not,

(i) what would make it complete?

(ii) How would this affect the two risk premia?

11.3. Let S_t denote the current price in dollars of one unit of foreign currency; $F_{t,T}$ is the delivery price agreed to in a forward contract; r is the domestic interest rate with continuous compounding; r^* is the foreign interest rate with continuous compounding.

(a) Consider the following pay-offs:

(i) investing in a domestic bond

(ii) investing a unit of domestic currency in a foreign bond and buying a forward contract to convert the proceeds.

Find the value of the forward exchange rate $F_{t,T}$.

(b) Suppose that the foreign interest rate exceeds the domestic interest rate at date t so that $r^* > r$. What is the relation between the forward and spot exchange rates?

11.4. (a) What is the price of a forward contract on a dividend-paying stock with stock price S_t?

(b) A one-year long forward contract on a non-dividend-paying stock is entered into when the stock price is \$40 and the risk-free interest rate is 10% per annum with continuous compounding. What is the forward price?

(c) Six months later, the price of the stock is \$45 and the risk-free interest rate is still 10%. What is the forward price?

11.5. Suppose that in an economy with one and two zero-coupon period bonds investors maximize $E_t \Sigma_{s=0}^{\infty} \beta^s \ln c_{t+s}$. What is

(a) the risk premium in period t for the two-period bond, and

(b) its price in period t?

(c) What is the forward rate for the two-period bond?

(d) Hence, express the risk premium in terms of this forward rate.

11.6. Consider a Vasicek model with two independent latent factors z_{1t} and z_{2t}. The price of an n-period bond and the log discount factor may be written as

$$\begin{aligned} p_{n,t} &= -[A_n + B_{1n} z_{1t} + B_{2n} z_{2t}] \\ m_{t+1} &= -[z_{1t} + z_{2t} + \lambda_1 e_{1,t+1} + \lambda_2 e_{2,t+1}] \end{aligned}$$

where the factors are generated by

$$z_{i,t+1} - \mu_i = \phi_i (z_{it} - \mu_i) + e_{i,t+1}, \quad i = 1, 2.$$

(a) Derive the no-arbitrage condition for an n-period bond and its risk premium. State any additional assumptions made.

(b) Explain how the yield on an n-period bond and its risk premium can be expressed in terms of the yields on one and two period bonds.

(c) Derive an expression for the n-period ahead forward rate.

(d) Comment on the implications of these results for the shape and behavior over time of the yield curve.

11.7 In their affine model of the term structure Ang and Piazzesi (2003) specify the pricing kernel M_t directly as follows:

$$
\begin{aligned}
M_{t+1} &= \exp(-s_t)\frac{\xi_{t+1}}{\xi_t} \\
s_t &= \delta_0 + \delta_1' z_t \\
z_t &= \mu + \phi' z_{t-1} + \Sigma e_{t+1} \\
\frac{\xi_{t+1}}{\xi_t} &= \exp(-\frac{1}{2}\lambda_t'\lambda_t - \lambda_t' e_{t+1}) \\
\lambda_t &= \lambda_0 + \lambda_1 z_t \\
p_{n,t} &= A_n + B_n' z_t
\end{aligned}
$$

(a) Derive the yield curve, and

(b) and the risk premium on a n-period yield.

Chapter 12

12.1. According to rational expectations models of the nominal exchange rate, such as the Monetary Model, an increase in the domestic money supply is expected to cause an appreciation in the exchange rate, but the exchange rate depreciates. Explain why the Monetary Model is nonetheless correct.

12.2 The Buiter-Miller (1981) model of the exchange rate - not formally a DGE model but, apart from the backward-looking pricing equation, broadly consistent with such an interpretation - may be represented as follows:

$$\begin{aligned}
y_t &= \alpha(s_t + p_t^* - p_t) - \beta(R_t - \Delta p_{t+1} - r_t) + g_t + \gamma y_t^* \\
m_t - p_t &= y_t - \lambda R_t \\
\Delta p_{t+1} &= \theta(y_t - y_t^n) + \pi_t^{\#} \\
\Delta s_{t+1} &= R_t - R_t^*
\end{aligned}$$

where y is output, y^n is full employment output, g is government expenditure, s is the log exchange rate, R is the nominal interest rate, r is the real interest rate, m is log nominal money, p is the log price level, $\pi^{\#}$ is target inflation and an asterisk denotes the foreign equivalent.

(a) Derive the long-run and short-run solutions for output, the price level and the exchange rate.

(b) Hence comment on the effects of monetary and fiscal policy.

(c) Suppose that the foreign country is identical and the two countries comprise the "world" economy. Denoting the corresponding world variable as $\overline{x}_t = x_t + x_t^*$ and the country differential by $\widetilde{x}_t = x_t - x_t^*$,

(i) derive the solutions for the world economy and for the differences between the economies.

(ii) Analyse the effects of monetary and fiscal policy on the world economy.

12.3 Consider a small cash-in-advance open economy with a flexible exchange rate in which output is exogenous, there is Calvo pricing, PPP holds in the long run, UIP holds and households maximize $\Sigma_{j=0}^{\infty}\beta^{j}\ln c_{t+j}$ subject to their budget constraint

$$S_t\Delta F_{t+1}+\Delta M_{t+1}+P_tc_t=P_tx_t+R_t^*S_tF_t$$

where P_t is the general price level, c_t is consumption, x_t is output, F_t is the net foreign asset position, M_t is the nominal money stock, S_t is the nominal exchange rate and R_t^* is the foreign nominal interest rate.

(a) Derive the steady-state solution of the model when output is fixed.

(b) Obtain a log-linear approximation to the model suitable for analysing its short-run behavior

(c) Comment on its dynamic properties.

12.4. Suppose the global economy consists of two identical countries who take output as given, have cash-in-advance demands for money based on the consumption of domestic and foreign goods and services, and who may borrow or save either through domestic or foreign bonds. Purchasing power parity holds and the domestic and foreign money supplies are exogenous. Global nominal bond holding satisfies $B_t+S_tB_t^*=0$ where S_t is the domestic price of nominal exchange and B_t is the nominal supply of domestic bonds. The two countries maximize $\Sigma_{j=0}^{\infty}\beta^{j}\ln c_{t+j}$ and $\Sigma_{j=0}^{\infty}\beta^{j}\ln c_{t+j}^*$, respectively, where c_t is real consumption. Foreign equivalents are denoted with an asterisk.

(a) Derive the solutions for consumption and the nominal exchange rate.

(b) What are the effects of increases in

(i) the domestic money supply and

(ii) domestic output?

12.5 Consider a world consisting of two economies A and B. Each produces a single tradeable good and issues a risky one-period bond with real rate of returns r_t^A and r_t^B, respectively. Noting that the real exchange rate e_t between these countries is the ratio of their marginal utilities,

(a) derive the real interest parity condition.

(b) How is this affected in the following cases:

(i) both countries are risk neutral,

(ii) markets are complete?

Chapter 13

13.1. Consider the following characterizations of the IS-LM and DGE models:

IS-LM

$$\begin{aligned} y &= c(y,r) + i(y,r) + g \\ m - p &= L(y,r) \end{aligned}$$

DGE

$$\begin{aligned} \Delta c &= -\frac{1}{\sigma}(r - \theta) = 0 \\ y &= c + i + g \\ y &= f(k) \\ \Delta k &= i \\ f_k &= r \end{aligned}$$

where y is output, c is consumption, i is investment, k is the capital stock, g is government expenditure, r is the real interest rate, m is log nominal money and p is the log price level.

(a) Comment on the main differences in the two models and on the underlying approaches to macroeconomics.

(b) Comment on the implications of the two models for the effectiveness of monetary and fiscal policy.

13.2. (a) How might a country's international monetary arrangements affect its conduct of monetary policy?

(b) What other factors might influence the way it carries out its monetary policy?

13.3. The Lucas-Sargent proposition is that systematic monetary policy is ineffective. Examine this hypothesis using the following model of the economy due to Bull and Frydman (1983):

$$\begin{aligned} y_t &= \alpha_1 + \alpha_2(p_t - E_{t-1}p_t) + u_t \\ d_t &= \beta(m_t - p_t) + v_t \\ \Delta p_t &= \theta(p_t^* - p_{t-1}), \end{aligned}$$

where y is output, d is aggregate demand, p is the log price level, p^* is the market clearing price, m is log nominal money and u and v are mean zero, mutually and serially independent shocks.

(a) Derive the solutions for output and prices.

(b) If $m_t = \mu + \varepsilon_t$ where ε_t is a mean zero serially independent shock, comment on the effect on prices of

(i) an unanticipated shock to money in period t,

(ii) a temporary anticipated shock to money in period t,

(iii) a permanent anticipated shock to money in period t.

(c) Hence comment on the Lucas-Sargent proposition.

13.4. Consider the following model of the economy:

$$\begin{aligned} x_t &= -\beta(R_t - E_t\pi_{t+1} - r) \\ \pi_t &= E_t\pi_{t+1} + \alpha x_t + e_t \\ R_t &= \gamma(E_t\pi_{t+1} - \pi^*), \end{aligned}$$

where π_t is inflation, π^* is target inflation, x_t is the output gap, R_t is the nominal interest rate and e_t is a mean zero serially independent shock.

(a) Why is the interest rate equation misspecified?

(b) Correct the specification and state the long-run solution.

(c) What are the short-run solutions for π_t, x_t and R_t?

(d) In the correctly specified model how would the behavior of inflation, output and monetary policy be affected by

(i) a temporary shock e_t

(ii) an expected shock e_{t+1}?

(e) Suppose that the output equation is modified to

$$x_t = -\beta(R_t - E_t\pi_{t+1} - r) - \theta e_t,$$

where e_t can be interpreted as a supply shock. How would the behavior of inflation, output and monetary policy be affected by a supply shock?

13.5 Consider the following New Keynesian model:

$$\begin{aligned}
\pi_t &= \phi + \beta E_t\pi_{t+1} + \gamma x_t + e_{\pi t} \\
x_t &= E_t x_{t+1} - \alpha(R_t - E_t\pi_{t+1} - \theta) + e_{xt} \\
R_t &= \theta + \pi^* + \mu(\pi_t - \pi^*) + \upsilon x_t + e_{Rt},
\end{aligned}$$

where π_t is inflation, π^* is target inflation, x_t is the output gap, R_t is the nominal interest rate, $e_{\pi t}$ and $e_{\pi t}$ are independent, zero-mean iid processes and $\phi = (1-\beta)\pi^*$.

(a) What is the long-run solution?

(b) Write the model in matrix form and obtain the short-run solutions for inflation and the output gap when $\dot{\mu} > 1$ and $\mu < 1$.

(c) Assuming the shocks are uncorrelated, derive the variance of inflation in each case and comment on how the choice of μ and ν affects the variance of inflation

(d) Hence comment on how to tell whether the "great moderation" of inflation in the early 2000's was due to good policy or to good fortune.

13.6. Consider the following model of Broadbent and Barro (1997):

$$\begin{aligned} y_t &= \alpha(p_t - E_{t-1}p_t) + e_t \\ d_t &= -\beta r_t + \varepsilon_t \\ m_t &= y_t + p_t - \lambda R_t \\ r_t &= R_t - E_t \Delta p_{t+1} \\ y_t &= d_t, \end{aligned}$$

where e_t and ε_t are zero-mean mutually and serially correlated shocks.

(a) Derive the solution to the model

(i) under money supply targeting,

(ii) inflation targeting,

(b) Derive the optimal money supply rule if monetary policy minimizes $E_t(p_{t+1} - E_t p_{t+1})^2$ subject to the model of the economy.

(c) What does this policy imply for inflation and the nominal interest rate?

(d) Derive the optimal interest rate rule.

(e) How would these optimal policies differ if monetary policy was based on targeting inflation instead of the price level?

13.7. Suppose that a monetary authority is a strict inflation targeter attempting to minimize $E(\pi_t - \pi^*)^2$ subject to the following model of the economy

$$\pi_t = \alpha_t R_t + z_t + e_t,$$

where $\alpha_t = \alpha + \varepsilon_t$, $E(z_t) = z + \varepsilon_{zt}$ and $\varepsilon_{\alpha t}$ and ε_{zt} are random measurement errors of α and z, respectively; $\varepsilon_{\alpha t}$, ε_{zt} and e_t are mutually and independently distributed random variables with zero means and variances σ_z^2, σ_α^2 and σ_e^2.

(a) What is the optimal monetary policy

(i) in the absence of measurement errors,

(ii) in the presence of measurement errors?

(b) What are broader implications of these results for monetary policy?

13.8. A highly stylized model of an open economy is

$$\begin{aligned} p_t &= \alpha p_{t-1} + \theta(s_t - p_t) \\ s_t &= R_t + R_{t+1}, \end{aligned}$$

where p_t is the price level, s_t is the exchange rate and R_t is the nominal interest rate. Suppose that monetary policy aims to choose R_t and R_{t+1} to minimize

$$L = (p_t - p^*)^2 + \beta(p_{t+1} - p^*)^2 + \gamma(R_t - R^*)^2,$$

where $p_{t-1} = R_{t+2} = 0$.

(a) Find the time consistent solutions for R_t and R_{t+1}. (Hint: first find R_{t+1} taking R_t as given.)

(b) Find the optimal solution by optimizing simultaneously with respect to R_t and R_{t+1}.

(c) Compare the two solutions and the significance of γ.

13.9. Consider the following model of an open economy:

$$\begin{aligned} \pi_t &= \mu + \beta E_t \pi_{t+1} + \gamma x_t + e_{\pi t} \\ x_t &= -\alpha(R_t - E_t \pi_{t+1} - \theta) + \phi(s_t + p_t^* - p_t) + e_{xt} \\ \Delta s_{t+1} &= R_t - R_t^* + e_{st}, \end{aligned}$$

where $e_{\pi t}$, e_{xt} and e_{st} are mean zero, mutually and serially independent shocks to inflation, output and the exchange rate.

(a) Derive the long-run solution making any additional assumptions thought necessary.

(b) Derive the short-run solution for inflation.

(c) Each period monetary policy is set to minimize $E_t(\pi_{t+1} - \pi^*)^2$, where π^* is the long-run solution for π, on the assumption that the interest rate chosen will remain unaltered indefinitely and foreign interest rate and price level will remain unchanged. Find the optimal value of R_t.

Chapter 14

14.1. Consider a variant on the basic real business cycle model. The economy is assumed to maximize $E_t \sum_{s=0}^{\infty} \beta^s \frac{c_{t+s}{}^{1-\sigma}}{1-\sigma}$ subject to

$$
\begin{aligned}
y_t &= c_t + i_t \\
y_t &= A_t k_t^{\alpha} \\
\Delta k_{t+1} &= i_t - \delta k_t \\
\ln A_t &= \rho \ln A_{t-1} + e_t,
\end{aligned}
$$

where y_t is output, c_t is consumption, i_t is investment, k_t is the capital stock, A_t is technical progress and $e_t \sim i.i.d(0, \omega^2)$.

(a) Derive

(i) the optimal short-run solution,

(ii) the steady-state solution,

(ii) a log-linearization to the short-run solution about its steady state in $\ln c_t - \ln c$ and $\ln k_t - \ln k$, where $\ln c$ and $\ln k$ are the steady-state values of $\ln c_t$ and $\ln k_t$.

(b) If, in practice, output, consumption and capital are non-stationary I(1) variables,

(i) comment on why this model is not a useful specification.

(ii) Suggest a simple re-specification of the model that would improve its usefulness.

(c) In practice, output, consumption and capital also have independent sources of random variation.

(i) Why is this not compatible with this model?

(ii) Suggest possible ways in which the model might be re-specified to achieve this.

14.2 After (log-) linearization all DSGE models can be written in the form

$$B_0 x_t = B_1 E_t x_{t+1} + B_2 z_t.$$

If there are lags in the model, then the equation will be in companion form and x_t and z_t will be long (state) vectors. And if B_0 is invertible then the DSGE model can also be written as

$$x_t = A_1 E_t x_{t+1} + A_2 z_t,$$

where $A_1 = B_0^{-1} B_1$ and $A_2 = B_0^{-1} B_2$.

(a) Show that the model in Exercise 14.1 can be written in this way.

(b) Hence show that the solution can be written as a vector autoregressive-moving average (VARMA) model.

(c) Hence comment on the effect of a technology shock.

14.3. Consider the real business cycle model defined in terms of the same variables as in Exercise 14.1 with the addition of employment, n_t:

$$\begin{aligned}
\mathcal{U}_t &= E_t \sum_{s=0}^{\infty} \beta^s [\frac{c_{t+s}{}^{1-\sigma}}{1-\sigma} - \gamma \frac{n_{t+s}{}^{1-\phi}}{1-\phi}] \\
y_t &= c_t + i_t \\
y_t &= A_t k_t^{\alpha} n_t^{1-\alpha} \\
\Delta k_{t+1} &= i_t - \delta k_t \\
\ln A_t &= \rho \ln A_{t-1} + e_t,
\end{aligned}$$

where $e_t \sim i.i.d(0, \omega^2)$.

(a) Derive the optimal solution

(b) Hence find the steady-state solution.

(c) Log-linearize the solution about its steady state to obtain the short-run solution.

(d) What is the implied dynamic behavior of the real wage and the real interest rate?

14.4 For a log-linearized version of the model of Exercise 14.1 write a Dynare program to compute the effect of an unanticipated temporary technology shock on the logarithms of output, consumption and capital and the implied real interest rate assuming that $\alpha = 0.33$, $\delta = 0.1$, $\sigma = 4$, $\theta = 0.05$, $\rho = 0.5$ and the variance of the technology shock e_t is zero.

Notes:

(i) Dynare runs in both Matlab and Gauss and is freely downloadable from http://www.dynare.org

(ii) Dynare uses a different dating convention. It dates non-jump variables like the capital stock at the end and not the start of the period, i.e. as $t-1$ and not t.

14.5 For the model of Exercise 14.1 write a Dynare program to compute the effect of a temporary technology shock assuming that $\alpha = 0.33$, $\delta = 0.1$, $\sigma = 4$, $\theta = 0.05$, $\rho = 0.5$ and the variance of the technology shock e_t is unity. Plot the impulse response functions for output, consumption, capital and the real interest rate.

14.6. For the model of Exercise 14.5 write a Dynare program for a stochastic simulation which calculates the means, variances, cross correlations and autocorrelations.

14.7 (a) Consider the New Keynesian model

$$\begin{aligned}
\pi_t &= \pi^* + \alpha(E_t\pi_{t+1} - \pi^*) + \beta(\pi_{t-1} - \pi^*) + \delta x_t + e_{\pi t} \\
x_t &= E_t x_{t+1} - \gamma(R_t - E_t\pi_{t+1} - \theta) + e_{xt} \\
R_t &= \theta + \pi^* + \mu(\pi_t - \pi^*) + \upsilon x_t,
\end{aligned}$$

where π_t is inflation, π^* is target inflation, x_t is the output gap, R_t is the nominal interest rate, $e_{\pi t}$ and e_{xt} are independent, shocks.ean iid processes and $\phi = (1-\beta)\pi^*$. Write a Dynare program to compute the effect of a supply shock in period t such that $e_{\pi t} = -e_{xt} = 5$. Assume that $\pi^* = 2$, $\alpha = 0.6$, $\alpha + \beta = 1$, $\delta = 1$, $\gamma = 5$, $\theta = 3$, $\mu = 1.5$ and $\nu = 1$.

(b) Compare the monetary policy response to the increase in inflation compared with that of a strict inflation targeter when $\nu = 0$.

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Solutions

Chapter2

2.1. We have assumed that the economy discounts s periods ahead using the geometric (or exponential) discount factor $\beta^s = (1+\theta)^{-s}$ for $\{s = 0, 1, 2, ...\}$. Suppose instead that the economy uses the sequence of hyperbolic discount factors $\beta_s = \{1, \varphi\beta, \varphi\beta^2, \varphi\beta^3, ...\}$ where $0 < \varphi < 1$.

(a) Compare the implications for discounting of using geometric and hyperbolic discount factors.

(b) For the centrally planned model

$$\begin{aligned} y_t &= c_t + i_t \\ \Delta k_{t+1} &= i_t - \delta k_t \end{aligned}$$

where y_t is output, c_t is consumption, i_t is investment, k_t is the capital stock and the objective is to maximize

$$V_t = \sum_{s=0}^{\infty} \beta_s U(c_{t+s})$$

derive the optimal solution under hyperbolic discounting and comment on any differences with the solution based on geometric discounting.

Solution

(a) As $0 < \varphi < 1$ the hyperbolic discount factor is smaller than the corresponding geometric discounting for a given value of β and $s > 0$. This implies that the future is discounted more and hence becomes less 'important' than before. Figure 2.1 illustrates the effect of different values of φ for a given value of $\beta = 0.9$.

(b) The Lagrangian with hyperbolic discounting is

$$\mathcal{L}_t = U(c_t) + \lambda_t[F(k_t) - c_t - k_{t+1} + (1-\delta)k_t] + \sum_{s=1}^{\infty}\{\varphi\beta^s U(c_{t+s}) + \lambda_{t+s}[F(k_{t+s}) - c_{t+s} - k_{t+s+1} + (1-\delta)k_{t+s}]\}$$

The first-order are conditions are

$$
\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} &= \begin{cases} U'(c_t) - \lambda_t = 0, & s = 0 \\ \varphi\beta^s U'(c_{t+s}) - \lambda_{t+s} = 0, & s > 0 \end{cases} \\
\frac{\partial \mathcal{L}_t}{\partial k_{t+s}} &= \lambda_{t+s}[F'(k_{t+s}) + 1 - \delta] - \lambda_{t+s-1} = 0, \qquad s > 0
\end{aligned}
$$

plus the resource constraints and the transversality condition $\lim_{s\to\infty} \varphi\beta^s U'(c_{t+s})k_{t+s} = 0$. The Euler equations for $s \geq 0$ can be written as

$$
\begin{aligned}
\varphi\beta\frac{U'(c_{t+1})}{U'(c_t)}[F'(k_{t+1}) + 1 - \delta] &= 1 && (1) \\
\beta\frac{U'(c_{t+s+1})}{U'(c_{t+s})}[F'(k_{t+s+1}) + 1 - \delta] &= 1, \qquad s > 0 && (2)
\end{aligned}
$$

These differ for the first period. Hence hyperbolic only differs from geometric discounting in the initial period. The long-run optimal levels of the capital stock and consumption will be the same as for geometric discounting. After period $t+1$ the short-run responses of consumption and capital to a shock will also be the same as for geometric discounting, but between periods t and $t+1$ the response of consumption along the optimal path differs.

2.2. Assuming hyperbolic discounting, the utility function $U(c_t) = \ln c_t$ and the production function $y_t = Ak_t$,

(a) derive the optimal long-run solution.

(b) Analyse the short-run solution.

Solution

(a) From the solution of exercise 1, equations (1) and (2) the optimal long run solution for capital is obtained from its net marginal product

$$F_k' - \delta = \alpha A k^{\alpha-1} = \theta$$

giving

$$
\begin{aligned}
k &= \left(\frac{\theta+\delta}{\alpha A}\right)^{-\frac{1}{1-\alpha}} \\
c &= Ak^{\alpha} - \delta k
\end{aligned}
$$

(b) The Euler equation implies that the optimal rate of growth in consumption from periods t to $t+1$ is

$$
\begin{aligned}
\frac{c_{t+1}}{c_t} &= \varphi\beta[F'(k_{t+1}) + 1 - \delta] \\
&= \varphi\beta(\alpha A k_{t+1}^{-(1-\alpha)} + 1 - \delta) \\
&< \beta(\alpha A k_{t+1}^{-(1-\alpha)} + 1 - \delta)
\end{aligned}
$$

the corresponding solution under geometric discounting.

To illustrate the different dynamic behavior of the economy consider a permanent productivity increase in period t. We have shown that the economy moves towards the same long-run solution as for geometric discounting. Moreover, from $t+1$, it approaches long-run equilibrium at the same speed. Although y_t is given in period t because k_t cannot be changed, c_t and i_t - and hence k_{t+1} - will be different from geometric discounting. As future consumption is less important, c_t will be higher, i_t and hence k_{t+1} must be lower. In other words, there is a bigger initial impact on consumption and thereafter consumption and the capital stock are smaller than for geometric discounting and it takes longer to reach the same long-run equilibrium.

2.3. Consider the CES production function $y_t = A[\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}]^{\frac{1}{1-\frac{1}{\gamma}}}$

(a) Show that the CES function becomes the Cobb-Douglas function as $\gamma \to 1$.

(b) Verify that the CES function is homogeneous of degree one and hence satisfies $F(k_t, n_t) = F_{n,t} n_t + F_{k,t} k_t$.

Solution

(a) We use L'Hospital's rule to obtain the solution. Consider re-writing the production function as

$$1 = \frac{y_t^{1-\frac{1}{\gamma}}}{A[\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha)n_t^{1-\frac{1}{\gamma}}]} = \frac{f(\gamma)}{g(\gamma)} = h(\gamma)$$

As $\gamma \to 1$, $1 - \frac{1}{\gamma} \to 0$ and $x_t^{1-\frac{1}{\gamma}} \to 1$. Hence $f(\gamma) \to 1$, $g(\gamma) \to A$ and $h(\gamma) \to \frac{1}{A} \neq 1$. Therefore the equation is violated as $\gamma \to 1$. L'Hospital's rule says that in a such a case we may obtain the limit as $\gamma \to 1$ using

$$1 = \frac{\lim_{\gamma\to 1} f'(\gamma)}{\lim_{\gamma\to 1} g'(\gamma)} = \frac{\ln y_t}{A[\alpha \ln k_t + (1-\alpha)\ln n_t]}$$

where we have used $x_t^{1-\frac{1}{\gamma}} = e^{(1-\frac{1}{\gamma})\ln x_t}$ and

$$\frac{de^{(1-\frac{1}{\gamma})\ln x_t}}{d\gamma} = -\frac{1}{\gamma^2} x_t^{1-\frac{1}{\gamma}} \ln x_t \to -\ln x_t \qquad \text{as } \gamma \to 1$$

Taking exponentials gives the Cobb-Douglas production function

$$y_t = e^A k_t^\alpha n_t^{1-\alpha}.$$

(b) First consider the marginal product of capital. The CES production function can be rewritten as

$$y_t^{1-\frac{1}{\gamma}} = A^{1-\frac{1}{\gamma}}[\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha)n_t^{1-\frac{1}{\gamma}}]$$

Hence, partially differentiating with respect to k_t, gives

$$(1-\frac{1}{\gamma})y_t^{-\frac{1}{\gamma}}\frac{\partial y_t}{\partial k_t} = A^{1-\frac{1}{\gamma}}\alpha(1-\frac{1}{\gamma})k_t^{-\frac{1}{\gamma}}$$

It follows that

$$\frac{\partial y_t}{\partial k_t} = \alpha A^{1-\frac{1}{\gamma}}\left(\frac{y_t}{k_t}\right)^{\frac{1}{\gamma}}$$

and

$$\frac{\partial y_t}{\partial n_t} = (1-\alpha)A^{1-\frac{1}{\gamma}}\left(\frac{y_t}{n_t}\right)^{\frac{1}{\gamma}}$$

If $F(k_t, n_t) = F_{n,t}n_t + F_{k,t}k_t$ holds then

$$\begin{aligned} F(k_t, n_t) &= F_{n,t}n_t + F_{k,t}k_t \\ &= \alpha A^{1-\frac{1}{\gamma}} \left(\frac{y_t}{k_t}\right)^{\frac{1}{\gamma}} k_t + (1-\alpha)A^{1-\frac{1}{\gamma}} \left(\frac{y_t}{n_t}\right)^{\frac{1}{\gamma}} n_t \\ &= A^{1-\frac{1}{\gamma}} y_t^{\frac{1}{\gamma}} [\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha)n_t^{1-\frac{1}{\gamma}}] \\ &= y_t. \end{aligned}$$

Hence

$$y_t = A[\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha)n_t^{1-\frac{1}{\gamma}}]^{\frac{1}{1-\frac{1}{\gamma}}}$$

2.4. Consider the following centrally-planned model with labor

$$\begin{aligned} y_t &= c_t + i_t \\ \Delta k_{t+1} &= i_t - \delta k_t \\ y_t &= A[\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha)n_t^{1-\frac{1}{\gamma}}]^{\frac{1}{1-\frac{1}{\gamma}}} \end{aligned}$$

where the objective is to maximize

$$V_t = \sum_{s=0}^{\infty} \beta^s [\ln c_{t+s} + \varphi \ln l_{t+s}], \quad \beta = \frac{1}{1+\theta}$$

where y_t is output, c_t is consumption, i_t is investment, k_t is the capital stock, n_t is employment and l_t is leisure ($l_t + n_t = 1$).

(a) Derive expressions from which the long-run solutions for consumption, labour and capital may be obtained.

(b) What are the implied long-run real interest rate and wage rate?

(c) Comment on the implications for labor of having an elasticity of substitution between capital and labor different from unity

(d) Obtain the long-run capital-labor ratio.

Solution

(a) This problem is special case of the analysis in Chapter 2 for particular specifications of the utility and production functions. The Lagrangian is

$$\begin{aligned}\mathcal{L}_t &= \sum_{s=0}^{\infty}\{\beta^s U(c_{t+s}, l_{t+s}) + \lambda_{t+s}[F(k_{t+s}, n_{t+s}) - c_{t+s} - k_{t+s+1} + (1-\delta)k_{t+s}] \\ &\quad +\mu_{t+s}[1 - n_{t+s} - l_{t+s}]\}\end{aligned}$$

which is maximized with respect to $\{c_{t+s}, l_{t+s}, n_{t+s}, k_{t+s+1}, \lambda_{t+s}, \mu_{t+s}; s \geq 0\}$. The first-order conditions are

$$\begin{aligned}\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} &= \beta^s U_{c,t+s} - \lambda_{t+s} = 0, \qquad s \geq 0 \\ \frac{\partial \mathcal{L}_t}{\partial l_{t+s}} &= \beta^s U_{l,t+s} - \mu_{t+s} = 0, \qquad s \geq 0 \\ \frac{\partial \mathcal{L}_t}{\partial n_{t+s}} &= \lambda_{t+s} F_{n,t+s} - \mu_{t+s} = 0, \qquad s \geq 0 \\ \frac{\partial \mathcal{L}_t}{\partial k_{t+s}} &= \lambda_{t+s}[F_{k,t+s} + 1 - \delta] - \lambda_{t+s-1} = 0, \qquad s > 0\end{aligned}$$

The consumption Euler equation for $s = 1$ is

$$\beta \frac{U_{c,t+1}}{U_{c,t}}[F_{k,t+1} + 1 - \delta] = 1$$

or, in this case,

$$\beta \frac{c_t}{c_{t+1}}[\alpha A^{1-\frac{1}{\gamma}} \left(\frac{y_{t+1}}{k_{t+1}}\right)^{\frac{1}{\gamma}} + 1 - \delta] = 1$$

The long-run static equilibrium solution is therefore

$$\frac{y}{k} = A^{1-\gamma}\left(\frac{\theta + \delta}{\alpha}\right)^{\gamma}$$

Eliminating λ_{t+s} and μ_{t+s} from the first-order conditions for consumption, leisure and employment gives for $s = 0$

$$\frac{\varphi}{l_t} = \frac{(1-\alpha)A^{1-\frac{1}{\gamma}}\left(\frac{y_t}{n_t}\right)^{\frac{1}{\gamma}}}{c_t} \tag{3}$$

Consequently, the long-run solution satisfies

$$\frac{c}{1-n} = \frac{1-\alpha}{\varphi}A^{1-\frac{1}{\gamma}}\left(\frac{y}{n}\right)^{\frac{1}{\gamma}} \tag{4}$$

The solutions for the long-run values c, k, l, n, and y are obtained by solving equations (3) and (4) simultaneously with

$$\begin{aligned}
y &= A[\alpha k^{1-\frac{1}{\gamma}} + (1-\alpha)n^{1-\frac{1}{\gamma}}]^{\frac{1}{1-\frac{1}{\gamma}}} \\
y &= c + \delta k \\
l + n &= 1
\end{aligned}$$

These equations form a non-linear system and do not have a closed-form solution.

(b) The implied long-run real wage is

$$\begin{aligned}
w_t &= F_{n,t} \\
&= (1-\alpha)A^{1-\frac{1}{\gamma}} \left(\frac{y_t}{n_t}\right)^{\frac{1}{\gamma}}
\end{aligned}$$

and the long-run real interest rate is $r = \theta$.

(c) The long-run demand for labor is

$$n = (1-\alpha)^{\gamma} A^{-(1-\gamma)} y w^{-\gamma}$$

Hence the greater the degree of substitutability between capital and labor γ, the more sensitive is the demand for labor to changes in the real wage. The labor share is

$$\frac{wn}{y} = (1-\alpha)^{\gamma} \left(\frac{w}{A}\right)^{1-\gamma}$$

Consequently, the smaller is the elasticity of substitution, the greater is the share of labor. If $\gamma = 1$ then the share of labor is constant. Note also that an increase in productivity A increases the share of labor if $\gamma > 1$.

(d) From the marginal products for capital and labor the long-run capital-labor ratio can be shown to be

$$\frac{k}{n} = \left[\frac{\alpha w}{(1-\alpha)(\theta+\delta)}\right]^{\gamma}$$

Thus, the higher the degree of substitutability γ, the greater is the response of the capital-labor ratio to changes in the rates of return on capital and labor.

2.5. (a) Comment on the statement: "the saddlepath is a knife-edge solution; once the economy departs from the saddlepath it is unable to return to equilibrium and will instead either explode or collapse."

(b) Show that although the solution for the basic centrally-planned economy of Chapter 2 is a saddlepath, it can be approximately represented by a stable autoregressive process.

Solution

(a) This statement reflects a misunderstanding of the nature of the dynamic solution to the DSGE model that often arises when a phase-diagram rather than an algebraic derivation of the solution is used. In fact, the economy cannot explode given the usual assumptions of the DSGE model such as the basic centrally-planned model of Chapter 2. The potential advantage of using a phase diagram is that it may be better able to represent non-linear dynamics than an algebraic solution which, due to its mathematical intractability, is commonly simplified by being formulated in terms of local deviations from long-run equilibrium.

The phase diagram depicting the dynamic behavior of the optimal solution for the basic centrally-planned economy in Chapter 2, Figure 10, is

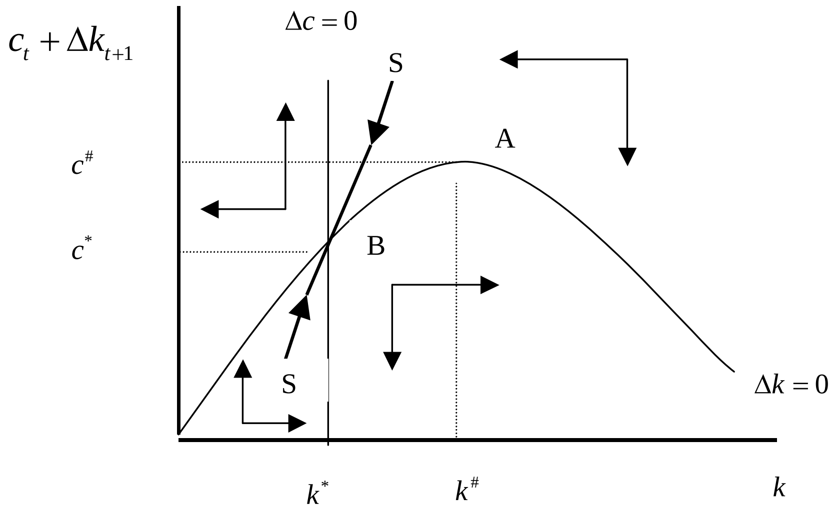


Figure 10

The arrows depict the dynamic behavior of consumption and capital for each point in space (for the positive orthant) and the saddlepath SS shows the saddlepath back to equilibrium at point B (i.e. the stable manifold). The functional form of the saddlepath is in general non-linear. It can also be approximated in the neighborhood of equilibrium (i.e. locally) by a partial adjustment mechanism with a forward-looking long-run solution based on current and expected future values of the exogenous variables. The dynamics of the adjustment to equilibrium following a shock are a first-order autoregressive process. Some examples are given in the Appendix, section 15.8.4. The functional form for the saddlepath and all its parameters are derived from the specification of the model's parameters. Change the model parameters and the saddlepath changes, but not its functional form. In contrast, the points in the phase diagram not on the saddlepath are not determined in terms of the model parameters. This is because, given the model, these points are unattainable. Only the points on the saddlepath are attainable. For this reason the arrows usually found in a phase may be misleading. They suggest that if the economy enters the wrong region it could explode or collapse. But, given the model, the economy can't enter such a region, and so cannot explode or collapse. Perhaps, therefore, there is less danger of misinterpreting the dynamic behavior of the economy if it is solving algebraically rather than being described using a phase diagram.

(b) For the basic centrally-planned economy of Chapter 2 we obtained the local approximation to the dynamic structure of the model as

$$\begin{bmatrix} c_{t+1} - c^* \\ k_{t+1} - k^* \end{bmatrix} = \begin{bmatrix} 1 + \frac{U'F''}{U''} & -(1+\theta)\frac{U'F''}{U''} \\ -1 & 1+\theta \end{bmatrix} \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix}$$

This can be rewritten as

$$z_{t+1} = Az_t + (I - A)z^*$$

where $z_t = (c_t, \ k_t)'$ and $z^* = (c^*, \ k^*)'$. The matrix A is a function of the parameters of the model, in particular, of the utility and production functions and of $c^*$and k^*. All of these are given. The two eigenvalues of A satisfy the saddlepath property that one is stable and the other is unstable. The eigenvalues were shown to be

$$\{\lambda_1, \ \lambda_2\} \simeq \{\frac{1+\theta}{2+\theta+\frac{U'F''}{U''}}, \ \ 2+\theta+\frac{U'F''}{U''} - \frac{1+\theta}{2+\theta+\frac{U'F''}{U''}}\}$$

with $0 < \lambda_1 < 1$ and $\lambda_2 > 1$.

A canonical factorization of A gives $A = Q^{-1}\Lambda Q$ where Q is a matrix of eigenvectors and Λ is a diagonal matrix of eigenvalues. Hence,

$$Qz_{t+1} = QAQ^{-1}Qz_t + Q(I - A)Q^{-1}Qz^*$$

or

$$w_{t+1} = \Lambda w_t + (I - \Lambda)w^*$$

where $w_t = (w_{1t}, \ w_{2t})' = Qz_t$ and $w^* = Qz^*$. Hence, we have two equations determining two variables:

$$\begin{aligned} w_{1,t+1} &= \lambda_1 w_{1t} + (1-\lambda_1)w_1^* \\ w_{2,t+1} &= \lambda_2 w_{2t} + (1-\lambda_2)w_2^* \end{aligned}$$

As $0 < \lambda_1 < 1$, the first equation shows that w_{1t} follows a stable autoregressive process. As $\lambda_2 > 1$, we rewrite the second equation as

$$w_{2t} = \lambda_2^{-1} w_{2,t+1} + (1 - \lambda_2^{-1}) w_2^*$$

and solve it forwards to give

$$\begin{aligned} w_{2t} &= (1 - \lambda_2^{-1}) \Sigma_{s=0}^{\infty} \lambda_2^{-s} w_2^* \\ &= w_2^* \end{aligned}$$

It follows that

$$w_{t+1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix} w_t + \begin{bmatrix} 1 - \lambda_1 & 0 \\ 0 & 1 \end{bmatrix} w^*$$

and so the solutions for consumption and capital are obtained from $z_t = Q^{-1} w_t$ or

$$z_{t+1} = Q^{-1} \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix} Q z_t + Q^{-1} \begin{bmatrix} 1 - \lambda_1 & 0 \\ 0 & 1 \end{bmatrix} Q z^*$$

Thus, the dynamic paths of consumption and capital (i.e. the saddlepath) may be approximated in the location of equilibrium by a stable first-order vector autoregression in which the parameters Q and λ_1 are obtained from A, and hence the model parameters.

2.6. In continuous time the basic centrally-planned economy problem can be written as: maximize $\int_0^\infty e^{-\theta t} u(c_t) dt$ with respect $\{c_t,\ k_t\}$ subject to the budget constraint $F(k_t) = c_t + \dot{k}_t + \delta k_t$.

(a) Obtain the solution using the Calculus of Variations.

(b) Obtain the solution using the Maximum Principle.

(c) Compare these solutions with the discrete-time solution of Chapter 2.

Solution

(a) The generic Calculus of Variations problem is concerned with choosing a path for x_t to maximize $\int_0^\infty f(x_t, \dot{x}_t, t) dt$ where $\dot{x}_t = \frac{dx_t}{dt}$ and x_t can be a vector. The first order conditions (i.e. the Euler equations) are $\frac{\partial f_t}{\partial x_t} - \frac{d}{dt}(\frac{\partial f_t}{\partial \dot{x}_t}) = 0$.

Consider the Lagrangian for this problem

$$\mathcal{L}_t = \int_0^\infty \{e^{-\theta t}u(c_t)dt + \lambda_t[F(k_t) - c_t - \dot{k}_t - \delta k_t]\}dt$$

where λ_t is the Lagrange multiplier. We therefore define

$$f(x_t, \dot{x}_t, t) = e^{-\theta t}u(c_t)dt + \lambda_t[F(k_t) - c_t - \dot{k}_t - \delta k_t]$$

and $x_t = \{c_t, k_t, \lambda_t\}'$. The first-order conditions are

$$\begin{aligned}
\frac{\partial f_t}{\partial c_t} - \frac{d}{dt}(\frac{\partial f_t}{\partial \dot{c}_t}) &= e^{-\theta t}u'(c_t) - \lambda_t = 0 \\
\frac{\partial f_t}{\partial k_t} - \frac{d}{dt}(\frac{\partial f_t}{\partial \dot{k}_t}) &= \lambda_t[F'(k_t) - \delta] + \dot{\lambda}_t = 0 \\
\frac{\partial f_t}{\partial \lambda_t} - \frac{d}{dt}(\frac{\partial f_t}{\partial \dot{\lambda}_t}) &= F(k_t) - c_t - \dot{k}_t - \delta k_t = 0
\end{aligned}$$

Noting that

$$\begin{aligned}
\dot{\lambda}_t &= -\theta e^{-\theta t}u'(c_t) + e^{-\theta t}\frac{d}{dt}[u'(c_t)] \\
&= e^{-\theta t}u'(c_t)\{\frac{\frac{d}{dt}[u'(c_t)]}{u'(c_t)} - \theta\} \\
&= \lambda_t\{\frac{\frac{d}{dt}[u'(c_t)]}{u'(c_t)} - \theta\} = 0,
\end{aligned}$$

by eliminating the Lagrange multipliers we may obtain from the second first-order condition the fundamental Euler equation

$$-\frac{\frac{d}{dt}[u'(c_t)]}{u'(c_t)} = F'(k_t) - \delta - \theta.$$

In steady-state, when c_t is constant, this gives the usual steady-state solution for the capital stock that $F'(k_t) = \delta + \theta$.

(b) The Maximum Principle is concerned with choosing $\{x_t,\ z_t\}$ to maximise $\int_0^\infty f(x_t, z_t, t)dt$ subject to the constraint $\dot{x}_t(= \frac{dx_t}{dt}) = g(x_t, z_t, t)$ by first defining the Hamiltonian function

$$h(x_t, z_t, \lambda_t) = f(x_t, z_t, t) + \lambda_t g(x_t, z_t, t)$$

The first order conditions are then

(i) $\frac{\partial h_t}{\partial x_t} = - \dot{\lambda}_t$

(ii) $\frac{\partial h_t}{\partial z_t} = 0$

(iii) $\frac{\partial h_t}{\partial \lambda_t} = \dot{x}_t$

The Hamiltonian for this problem is

$$h(x_t, z_t, \lambda_t) = e^{-\theta t} u(c_t) + \lambda_t [F(k_t) - c_t - \delta k_t]$$

where $x_t = k_t$ and $z_t = c_t$. The first-order conditions are therefore

$$\begin{aligned}
\text{(i) } \frac{\partial h_t}{\partial k_t} &= \lambda_t [F'(k_t) - \delta] = -\dot{\lambda}_t \\
\text{(ii) } \frac{\partial h_t}{\partial c_t} &= e^{-\theta t} u'(c_t) - \lambda_t = 0 \\
\text{(iii) } \frac{\partial h_t}{\partial \lambda_t} &= F(k_t) - c_t - \delta k_t = \dot{k}_t
\end{aligned}$$

These are identical to those we derived using the Calculus of Variations and so we obtain exactly the same solution.

(c) We have already noted that the steady-state solution for the capital stock is the same in continuous and discrete time. Comparing the Euler equations. For the discrete case

$$\beta \frac{U_{c,t+1}}{U_{c,t}} [F_{k,t+1} + 1 - \delta] = 1.$$

Hence

$$\beta [1 + \frac{\Delta U_{c,t+1}}{U_{c,t}}][F_{k,t+1} + 1 - \delta] = 1$$

or

$$-\frac{\Delta U_{c,t+1}}{U_{c,t}} = \frac{F_{k,t+1} - \delta - \theta}{F_{k,t+1} + 1 - \delta}$$

which may be compared with $-\frac{\frac{d}{dt}[u'(c_t)]}{u'(c_t)} = F'(k_t) - \delta - \theta$. Apart from the discrete change instead of the continuous derivative on the left-hand side (where the limit is taken from above), the differences on the right-hand side are the timing of capital and the presence of the demoninator. As $F_{k,t+1} - \delta = r_{t+1}$, the net rate of return to capital, and this is θ in the steady-state, the denominator is approximately $1 + \theta$. This reveals that it is the different way of discounting in continuous from discrete time, together with the treatment of time itself, that are the main causes of the difference between the two solutions.

Chapter 3

3.1. Re-work the optimal growth solution in terms of the original variables, i.e. without first taking deviations about trend growth.

(a) Derive the Euler equation

(b) Discuss the steady-state optimal growth paths for consumption, capital and output.

Solution

(a) The problem is to maximize

$$\sum_{s=0}^{\infty} \beta^s U(C_{t+s})$$

where $U(C_t) = \frac{C_t^{1-\sigma}-1}{1-\sigma}$, subject to the national income identity, the capital accumulation equation, the production function and the growth of population n:

$$\begin{aligned}
Y_t &= C_t + I_t \\
\Delta K_{t+1} &= I_t - \delta K_t \\
Y_t &= (1+\mu)^t K_t^{\alpha} N_t^{1-\alpha} \\
N_t &= (1+n)^t N_0, \qquad N_0 = 1
\end{aligned}$$

The Lagrangian for this problem written in terms of the original variables is

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \{\beta^s \left[\frac{C_{t+s}^{1-\sigma} - 1}{1-\sigma}\right] + \lambda_{t+s}[\phi^{t+s} K_{t+s}^{\alpha} - C_{t+s} - K_{t+s+1} + (1-\delta)K_{t+s}]\}$$

where $\phi = (1+\mu)(1+n)^{(1-\alpha)} \simeq (1+\eta)^{1-\alpha}$, $\eta = n + \frac{\mu}{1-\alpha}$. The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial C_{t+s}} &= \beta^s C_{t+s}^{-\sigma} - \lambda_{t+s} = 0 \qquad && s \geq 0 \\
\frac{\partial \mathcal{L}_t}{\partial K_{t+s}} &= \lambda_{t+s}[\alpha\phi^{t+s} K_{t+s}^{\alpha-1} + 1 - \delta] - \lambda_{t+s-1} = 0 \qquad && s > 0
\end{aligned}$$

Hence the Euler equation is

$$\beta(\frac{C_{t+1}}{C_t})^{-\sigma}[\alpha\phi^{t+1} K_{t+1}^{\alpha-1} + 1 - \delta] = 1$$

(b) The advantage of transforming the variables as in Chapter 3 is now apparent. It enabled us to derive the steady-state solution in a similar way to static models and hence to use previous results. Now we need a different approach. If we assume that in steady state consumption grows at an arbitrary constant rate γ, then the Euler equation can be rewritten

$$\beta(1+\gamma)^{-\sigma}[\alpha\phi^{t+1}K_{t+1}^{\alpha-1}+1-\delta]=1$$

Hence the steady-state path of capital is

$$\begin{aligned} K_t &= \psi^{-\frac{1}{1-\alpha}}\phi^{\frac{1}{1-\alpha}t} \\ &\simeq \psi^{-\frac{1}{1-\alpha}}(1+\eta)^t \end{aligned}$$

where $\psi = \frac{(1+\theta)(1+\gamma)^{\sigma}+\delta-1}{\alpha}$. Hence, in steady state, capital grows approximately at the rate $\eta = n + \frac{\mu}{1-\alpha}$ as before.

The production function in steady state is

$$\begin{aligned} Y_t &= \phi^t K_t^{\alpha} = \psi^{-\frac{\alpha}{1-\alpha}}\phi^{\frac{1}{1-\alpha}t} \\ &\simeq \psi^{-\frac{\alpha}{1-\alpha}}(1+\eta)^t \end{aligned}$$

Thus output also grows at the rate η.

The resource constraint for the economy is

$$Y_t = C_t + \Delta K_{t+1} - \delta K_t$$

In steady state this becomes

$$\psi^{-\frac{\alpha}{1-\alpha}}\phi^{\frac{1}{1-\alpha}t} = C_t + (\phi^{\frac{1}{1-\alpha}}-1)\psi^{-\frac{1}{1-\alpha}}\phi^{\frac{1}{1-\alpha}t} - \delta\psi^{-\frac{1}{1-\alpha}}\phi^{\frac{1}{1-\alpha}t}$$

Hence steady-state consumption is

$$\begin{aligned} C_t &= (2-\phi^{\frac{1}{1-\alpha}}+\delta)\psi^{-\frac{\alpha}{1-\alpha}}\phi^{\frac{1}{1-\alpha}t} \\ &\simeq (2-\phi^{\frac{1}{1-\alpha}}+\delta)\psi^{-\frac{\sigma}{1-\alpha}}(1+\eta)^t \end{aligned}$$

implying that consumption grows at the same constant rate as capital and output, which confirms our original assumption and shows that $\gamma = \eta$. We recall that, as the growth rates of output, capital and consumption are the same, the optimal solution is a balanced growth path.

3.2. Consider the Solow-Swan model of growth for the constant returns to scale production function $Y_t = F[e^{\mu t}K_t, e^{\nu t}N_t]$ where μ and ν are the rates of capital and labor augmenting technical progress.

(a) Show that the model has constant steady-state growth when technical progress is labor augmenting.

(b) What is the effect of the presence of non-labor augmenting technical progress?

Solution

(a) First we recall some key results from Chapter 3. The savings rate for the economy is $s_t = 1 - \frac{C_t}{Y_t} = i_t$, the rate of investment I_t/Y_t. The rate of growth of population is n and of capital is $\frac{\Delta K_{t+1}}{K_t} = s\frac{y_t}{k_t} - \delta$; the growth of capital per capita is $\frac{\Delta k_{t+1}}{k_t} = s\frac{y_t}{k_t} - (\delta + n)$ and the capital accumulation equation is $\Delta k_{t+1} = sy_t - (\delta + n)k_t$ where $y_t = Y_t/N_t$ and $k_t = K_t/N_t$. Hence the sustainable rate of growth of capital per capita is

$$\gamma = \frac{\Delta k_{t+1}}{k_t} = s\frac{y_t}{k_t} - (\delta + n)$$

For the given production function

$$\frac{y_t}{k_t} = e^{\mu t}F[1, e^{(\nu-\mu)t}k_t^{-1}] = e^{\mu t}G[e^{(\nu-\mu)t}k_t^{-1}]$$

and so

$$\gamma = se^{\mu t}G[e^{(\nu-\mu)t}k_t^{-1}] - (\delta + n)$$

For the rate of growth of capital to be constant we therefore require that $\frac{y_t}{k_t}$ is constant. If $\mu = 0$, and hence technical progress is solely labor augmenting, then we simply require that $k_t = e^{\nu t}$. The rate of growth of capital is then $\nu + n$.

(b) If $\mu \neq 0$ then we have non-labor augmenting technical progress too. Consequently, in general, we then do not obtain a constant steady-state rate of growth. For the rate of growth of capital per capita γ to be constant we require that the function $G[.]$ satisfies $e^{\mu t}G[e^{(\nu-\mu-\gamma)t}]$ being constant. If the production function is homogeneous of degree one then this condition holds and non-labor augmenting technical progress would be consistent with steady-state growth. For an example see the next exercise.

3.3. Consider the Solow-Swan model of growth for the production function $Y_t = A(e^{\mu t}K_t)^{\alpha}(e^{\nu t}N_t)^{\beta}$ where μ is the rate of capital augmenting technical progress and ν is the rate of labor augmenting technical progress. Consider whether a steady-state growth solution exists under

(a) increasing returns to scale, and

(b) constant returns to scale.

(c) Hence comment on the effect of the degree of returns to scale on the rate of economic growth, and the necessity of having either capital or labor augmenting technical progress in order to achieve economic growth.

Solution

(a) Increasing returns to scale occurs if $\alpha+\beta > 1$. We use the same notation as in the previous exercise, and note that if a steady-state solution exists, then the sustainable rate of growth of capital per capita must satisfy

$$\gamma = \frac{\Delta k_{t+1}}{k_t} = s\frac{y_t}{k_t} - (\delta + n)$$

It follows from the production function and the growth of population $N_t = e^{nt}$ that

$$\frac{y_t}{k_t} = Ae^{[\alpha\mu+\beta\nu+(\alpha+\beta-1)n]t}k_t^{-(1-\alpha)}$$

hence

$$\gamma = sAe^{[\alpha\mu+\beta\nu+(\alpha+\beta-1)n]t}k_t^{-(1-\alpha)} - (\delta + n)$$

For the rate of growth of capital per capita to be constant we therefore require that

$$k_t = e^{\frac{\alpha\mu+\beta\nu+(\alpha+\beta-1)n}{1-\alpha}t}$$

i.e. that

$$\gamma = \frac{\alpha\mu + \beta\nu + (\alpha+\beta-1)n}{1-\alpha}$$

Consequently, a steady-state solution exists for any $\alpha + \beta > 0$ or any μ, $\nu > 0$, and this also satisfies

$$\gamma = sA - (\delta + n)$$

which is constant.

(b) If there are constant returns to scale $\alpha + \beta = 1$. In this case

$$\gamma = sAe^{[\alpha\mu+(1-\alpha)\nu]t}k_t^{-(1-\alpha)} - (\delta + n)$$

For the rate of growth of capital per capita to be constant we therefore require that

$$k_t = e^{\frac{\alpha\mu+(1-\alpha)\nu}{1-\alpha}t}$$

i.e. that

$$\gamma = \frac{\alpha\mu + (1-\alpha)\nu}{1-\alpha}$$

(c) Comparing the two cases we note that, if there is no technical progress, so that $\mu = \nu = 0$, then, with non-constant returns to scale, the rate of growth is

$$\gamma = \frac{(\alpha+\beta-1)n}{1-\alpha}$$

Hence, $\gamma \gtreqless 0$ as $\alpha + \beta \gtreqless 1$. In other words, even without technical progress, there is positive growth if returns to scale are increasing but, if there are constant returns to scale, then $\gamma = 0$ and so technical progress is required to achieve growth.

The results in Exercises 3.2 and 3.3 also hold for optimal growth which, in effect, simply adds the determination of the savings rate to the Solow-Swan model. In steady-state growth this is constant.

(ii) We have also shown that with a Cobb-Douglas production function it is possible to achieve steady-state growth with a mixture of capital and labor augmenting technical progress, or with just capital and no labor augmenting technical progress.

3.4. Consider the following two-sector endogenous growth model of the economy due to Rebelo (1991) which has two types of capital, physical k_t and human h_t. Both types of capital are required to produce goods output y_t and new human capital i_t^h. The model is

$$\begin{aligned} y_t &= c_t + i_t^k \\ \Delta k_{t+1} &= i_t^k - \delta k_t \\ \Delta h_{t+1} &= i_t^h - \delta h_t \\ y_t &= A(\phi k_t)^\alpha (\mu h_t)^{1-\alpha} \\ i_t^h &= A[(1-\phi)k_t]^\varepsilon [(1-\mu)h_t]^{1-\varepsilon} \end{aligned}$$

where i_t^k is investment in physical capital, ϕ and μ are the shares of physical and human capital used in producing goods and $\alpha > \varepsilon$. The economy maximizes $V_t = \Sigma_{s=0}^{\infty} \beta^s \frac{c_{t+s}^{1-\sigma}}{1-\sigma}$.

(a) Assuming that each type of capital receives the same rate of return in both activities, find the steady-state ratio of the two capital stocks

(b) Derive the optimal steady-state rate of growth.

(c) Examine the special case of $\varepsilon = 0$.

Solution

(a) Eliminating y_t, i_t^k and i_t^h, the Lagrangian for this problem may be written as

$$\begin{aligned}\mathcal{L}_t &= \Sigma_{s=0}^{\infty}\{\beta^s \frac{c_{t+s}^{1-\sigma}}{1-\sigma} + \lambda_{t+s}[A(\phi k_{t+s})^{\alpha}(\mu h_{t+s})^{1-\alpha} - c_{t+s} - k_{t+s+1} + (1-\delta)k_{t+s}] \\ &\quad + \gamma_{t+s}[A[(1-\phi)k_{t+s}]^{\varepsilon}[(1-\mu)h_{t+s}]^{1-\varepsilon} - h_{t+s+1} + (1-\delta)h_{t+s}]\}\end{aligned}$$

This must be maximized with respect to c_{t+s}, k_{t+s}, h_{t+s} and the Lagrange multipliers λ_{t+s} and γ_{t+s}. The first-order conditions are

$$\begin{aligned}\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} &= \beta^s c_{t+s}^{-\sigma} - \lambda_{t+s} = 0 \qquad\qquad s \geq 0 \\ \frac{\partial \mathcal{L}_t}{\partial k_{t+s}} &= \lambda_{t+s}[\alpha\phi A(\frac{\phi k_{t+s}}{\mu h_{t+s}})^{-(1-\alpha)} + 1 - \delta] - \lambda_{t+s-1} + \gamma_{t+s}\varepsilon(1-\phi)A[\frac{(1-\phi)k_{t+s}}{(1-\mu)h_{t+s}}]^{-(1-\varepsilon)} = 0,\ s > 0 \\ \frac{\partial \mathcal{L}_t}{\partial h_{t+s}} &= \lambda_{t+s}(1-\alpha)\mu A(\frac{\phi k_{t+s}}{\mu h_{t+s}})^{\alpha} + \gamma_{t+s}\{(1-\varepsilon)(1-\mu)A[\frac{(1-\phi)k_{t+s}}{(1-\mu)h_{t+s}}]^{\varepsilon} + 1 - \delta\} - \gamma_{t+s-1} = 0,\ s > 0\end{aligned}$$

Consider $s = 1$. As each type of capital receives the same rate of return, r_{t+1} say, then the net marginal products for the two types of capital satisfy

$$\begin{aligned}\alpha\phi A\left[\frac{\phi k_{t+1}}{\mu h_{t+1}}\right]^{-(1-\alpha)} - \delta &= r_{t+1} \\ (1-\varepsilon)(1-\mu)A\left[\frac{(1-\phi)k_{t+1}}{(1-\mu)h_{t+1}}\right]^{\varepsilon} - \delta &= r_{t+1}\end{aligned}$$

and r_{t+1} will be constant. From the rates of return, the steady-state ratio of the capital stocks is

$$\frac{k}{h} = \left[\frac{\alpha\phi(\frac{\phi}{\mu})^{-(1-\alpha)}[\frac{1-\phi}{1-\mu}]^{-\varepsilon}}{(1-\varepsilon)(1-\mu)}\right]^{\frac{1}{1+\varepsilon-\alpha}}$$

(b) If a steady-state solution exists then c_t, k_t, h_t all grow at the same rate η, say. From the first first-order condition $\lambda_{t+1} = \beta(1+\eta)^{-\sigma}\lambda_t$ and $\gamma_{t+1} = \beta(1+\eta)^{-\sigma}\gamma_t$. The last two first-order conditions can be written as the system of equations

$$\begin{aligned}a\lambda_t + b\gamma_t &= 0 \\ c\lambda_t + a\gamma_t &= 0\end{aligned}$$

where a, b and c are the constants

$$\begin{aligned} a &= \beta(1+r) - (1+\eta)^{\sigma} \\ b &= \beta\varepsilon(1-\phi)A^{\frac{1}{\varepsilon}}\left[\frac{r+\delta}{(1-\varepsilon)(1-\mu)}\right]^{-\frac{1-\varepsilon}{\varepsilon}} \\ c &= \beta(1-\alpha)\mu A^{\frac{1}{1-\alpha}}\left[\frac{r+\delta}{\alpha\phi}\right]^{-\frac{\alpha}{1-\alpha}} \end{aligned}$$

Hence,

$$\frac{\lambda_t}{\gamma_t} = -\frac{b}{a} = -\frac{a}{c}$$

or $a = \pm\sqrt{bc}$. Only a depends on η and assuming that $\beta = \frac{1}{1+\theta}$ we have $a \simeq r - \theta - \sigma\eta$. Hence η, the steady-state rate of growth of c_t, k_t, h_t, is

$$\eta \simeq \frac{1}{\sigma}[r - \theta - a]$$

As a is proportional to $(r+\delta)^{-[\frac{\alpha}{1-\alpha}+\frac{1-\varepsilon}{\varepsilon}]}$, we can express η as a function just of $\frac{k}{h}$; a decrease in $\frac{k}{h}$ would raise η.

(c) If $\varepsilon = 0$ then k_t is not required in the production of h_t. The consumption Euler equation is then the familiar expression

$$\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}(1+r_{t+1}) = 1$$

or

$$\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}(\alpha\phi A(\frac{\phi k_{t+1}}{\mu h_{t+1}})^{-(1-\alpha)} + 1 - \delta) = 1$$

Thus, the steady-state rate of growth of consumption is

$$\begin{aligned} \eta &= \left(\frac{1+r}{1+\theta}\right)^{\frac{1}{\sigma}} - 1 \\ &\simeq \frac{1}{\sigma}(r-\theta) \end{aligned}$$

As $\beta = \frac{1}{1+\theta}$, then if private saving continues until its rate of return equals that of private capital k, the steady-state growth rate is zero. In contrast, in part (b) due to human capital accumulation, we can have economic growth even if $r = \theta$.

Chapter 4

4.1. The household budget constraint may be expressed in different ways from equation (4.2) where the increase in assets from the start of the current to the next period equals total income less consumption. Derive the Euler equation for consumption and compare this with the solution based on equation (4.2) for each of the following ways of writing the budget constraint:

(a) $a_{t+1} = (1+r)(a_t + x_t - c_t)$, i.e. current assets and income assets that are not consumed are invested.

(b) $\Delta a_t + c_t = x_t + ra_{t-1}$, where the dating convention is that a_t denotes the end of period stock of assets and c_t and x_t are consumption and income during period t.

(c) $W_t = \Sigma_{s=0}^{\infty} \frac{c_{t+s}}{(1+r)^s} = \Sigma_{s=0}^{\infty} \frac{x_{t+s}}{(1+r)^s} + (1+r)a_t$, where W_t is household wealth.

Solution

(a) The Lagrangian is now defined as

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \{\beta^s U(c_{t+s}) + \lambda_{t+s}[(1+r)(a_{t+s} + x_{t+s} - c_{t+s}) - a_{t+s+1}]\}$$

The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} &= \beta^s U'(c_{t+s}) - \lambda_{t+s}(1+r) = 0 \qquad s \geq 0 \\
\frac{\partial \mathcal{L}_t}{\partial a_{t+s}} &= \lambda_{t+s}(1+r) - \lambda_{t+s-1} = 0 \qquad s > 0
\end{aligned}$$

Once again, eliminating λ_{t+s} and setting $s = 1$ gives the Euler equation

$$\frac{\beta U'(c_{t+1})}{U'(c_t)}(1+r) = 1$$

(b) The Lagrangian is now

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \{\beta^s U(c_{t+s}) + \lambda_{t+s}[x_{t+s} + (1+r)a_{t+s-1} - c_{t+s} - a_{t+s}]\}$$

The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} &= \beta^s U'(c_{t+s}) - \lambda_{t+s} = 0 \qquad s \geq 0 \\
\frac{\partial \mathcal{L}_t}{\partial a_{t+s}} &= \lambda_{t+s+1}(1+r) - \lambda_{t+s} = 0 \qquad s > 0
\end{aligned}$$

Eliminating λ_{t+s} and setting $s = 1$ again gives the Euler equation $\frac{\beta U'(c_{t+1})}{U'(c_t)}(1 + r) = 1$.

(c) In this case the Lagrangian only has a single constraint, and not a constraint for each period. It can be written as

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \beta^s U(c_{t+s}) + \lambda_t[\sum_{s=0}^{\infty} \frac{x_{t+s} - c_{t+s}}{(1 + r)^s} + (1 + r)a_t]\}$$

As a_t is given, we only require the first-order condition for c_{t+s}, which is

$$\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} = \beta^s U'(c_{t+s}) - \lambda_t \frac{1}{(1 + r)^s} = 0 \qquad s \geq 0$$

Combining the first-order conditions for $s = 0$ and $s = 1$ enables us to eliminate λ_t to obtain

$$\lambda_t = \beta U'(c_{t+1})(1 + r) = U'(c_t)$$

or $\frac{\beta U'(c_{t+1})}{U'(c_t)}(1 + r) = 1$, the same Euler equation as before.

We have shown, therefore, that expressing the household's problem in any of these three alternative ways produces the same Euler equation as in Chapter 4.

4.2. The representative household is assumed to choose $\{c_t, c_{t+1},...\}$ to maximise $V_t = \sum_{s=0}^{\infty} \beta^s U(c_{t+s})$, $0 < \beta = \frac{1}{1+\theta} < 1$ subject to the budget constraint $\Delta a_{t+1} + c_t = x_t + r_t a_t$ where c_t is consumption, x_t is exogenous income, a_t is the (net) stock of financial assets at the beginning of period t and r is the (constant) real interest rate.

(a) Assuming that $r = \theta$ and using the approximation $\frac{U'(c_{t+1})}{U'(c_t)} = 1 - \sigma \Delta \ln c_{t+1}$, show that optimal consumption is constant.

(b) Does this mean that changes in income will have no effect on consumption? Explain.

Solution

(a) In view of the previous exercise we go straight to the Euler equation which we write as

$$\frac{\beta U'(c_{t+1})}{U'(c_t)}(1 + r) = 1$$

Using the approximation $\frac{U'(c_{t+1})}{U'(c_t)} = 1 - \sigma \Delta \ln c_{t+1}$ and noting that $r = \theta$, the future rate of growth of consumption along the optimal path is

$$\Delta \ln c_{t+1} = \frac{r - \theta}{\sigma} = 0$$

i.e. optimal consumption is constant.

(b) Strictly, this result should be interpreted as saying that the future optimal level of consumption is the same as the curent level of consumption if current and future income are correctly anticipated in period t. In other words, correctly anticipated income has no effect on consumption beyond that already incorporated in current consumption c_t. However, if future income is not correctly anticipated then this result no longer holds. In order to analyse this case we should take explicit account of the role of expectations; so far we have ignored the distinction. Thus, we should express the above result as (see Chapter 10)

$$E_t c_{t+1} = c_t$$

From the inter-temporal budget constraint, in period t expected optimal consumption is

$$c_t = r E_t \sum_{s=0}^{\infty} \frac{x_{t+s}}{(1+r)^{s+1}} + r a_t$$

and c_{t+1} is expected to be

$$E_t c_{t+1} = r E_t \sum_{s=0}^{\infty} \frac{x_{t+s+1}}{(1+r)^{s+1}} + r E_t a_{t+1}$$

But in fact it is

$$c_{t+1} = E_{t+1} c_{t+1} = r E_{t+1} \sum_{s=0}^{\infty} \frac{x_{t+s+1}}{(1+r)^{s+1}} + r a_{t+1}$$

Hence,

$$c_{t+1} - E_t c_{t+1} = r \sum_{s=0}^{\infty} \frac{E_{t+1} x_{t+s+1} - E_t x_{t+s+1}}{(1+r)^{s+1}} + r(a_{t+1} - E_t a_{t+1})$$

Because a_{t+1} is given at the start of period $t+1$, we have $a_{t+1} - E_t a_{t+1} = 0$. Thus, any change in the expectation of income between periods t and $t+1$ would have an effect on consumption in period $t+1$ - and beyond. In particular, consider an unanticipated increase in income solely in period $t+1$ so that $E_t x_{t+1} \neq E_{t+1} x_{t+1} = x_{t+1}$. It follows that

$$c_{t+1} - E_t c_{t+1} = \frac{r}{1+r}(x_{t+1} - E_t x_{t+1}) \neq 0$$

even though we still have $c_{t+1} - E_t c_{t+1} = c_{t+1} - c_t$. In other words, income can affect optimal consumption; for example, an unanticipated change in future in income will affect consumption.

4.3 (a) Derive the dynamic path of optimal household consumption when the utility function reflects exogenous habit persistence h_t and the utility function is $U(c_t) = \frac{(c_t - h_t)^{1-\sigma}}{1-\sigma}$. and household budget constraint is $\Delta a_{t+1} + c_t = x_t + r a_t$.

(b) Hence, obtain the consumption function making the assumption that $\beta(1+r) = 1$. Comment on the case where expected future levels of habit persistence are the same as those in the current period, i.e. $h_{t+s} = h_t$ for $s \geq 0$.

Solution

(a) From Exercise 4.1 the Euler equation for this problem is

$$\beta \left[\frac{c_{t+1} - h_{t+1}}{c_t - h_t} \right]^{-\sigma} (1+r) = 1$$

or

$$c_{t+1} - h_{t+1} = [\beta(1+r)]^{-\frac{1}{\sigma}} (c_t - h_t)$$

Hence consumption follows a simple difference equation.

(b) To obtain the consumption function we solve the budget constraint forwards to get

$$a_t = \Sigma_{s=0}^{\infty} \frac{c_{t+s} - x_{t+s}}{(1+r)^{s+1}}$$

Hence,

$$\begin{aligned} a_t &= \Sigma_{s=0}^{\infty} \frac{[x_{t+s} - h_{t+s}] - [c_{t+s} - h_{t+s}]}{(1+r)^{s+1}} \\ &= \Sigma_{s=0}^{\infty} \frac{x_{t+s} - h_{t+s}}{(1+r)^{s+1}} - \frac{1}{r}(c_t - h_t) \end{aligned}$$

The consumption function is therefore

$$c_t = h_t + r\Sigma_{s=0}^{\infty}\frac{x_{t+s} - h_{t+s}}{(1+r)^s} + ra_t$$

If $h_{t+s} = h_t$ for $s \geq 0$, then this reduces to the standard consumption function

$$c_t = \frac{r}{1+r}\Sigma_{s=0}^{\infty}\frac{x_{t+s}}{(1+r)^s} + ra_t.$$

4.4. Derive the behavior of optimal household consumption when the utility function reflects habit persistence of the following forms:

(a) $U(c_t) = -\frac{(c_t - \gamma c_{t-1})^2}{2} + \alpha(c_t - \gamma c_{t-1})$

(b) $U(c_t) = \frac{(c_t - \gamma c_{t-1})^{1-\sigma}}{1-\sigma}$.

where the budget constraint is $\Delta a_{t+1} + c_t = x_t + ra_t$.

(c) Compare (b) with the case where $U(c_t) = \frac{(c_t - h_t)^{1-\sigma}}{1-\sigma}$ and h_t is an exogenous habitual level of consumption.

Solution

(a) The Lagrangian is

$$\mathcal{L}_t = \sum_{s=0}^{\infty}\{\beta^s[-\frac{(c_{t+s} - \gamma c_{t+s-1})^2}{2} + \alpha(c_{t+s} - \gamma c_{t+s-1})] + \lambda_{t+s}[x_{t+s} + (1+r)a_{t+s} - c_{t+s} - a_{t+s+1}]\}$$

The first-order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}_t}{\partial c_{t+s}} &= -\beta^s[\alpha - (c_{t+s} - \gamma c_{t+s-1})] - \beta^{s+1}\gamma[\alpha - (c_{t+s+1} - \gamma c_{t+s})] - \lambda_{t+s} = 0 \qquad s \geq 0 \\ \frac{\partial \mathcal{L}_t}{\partial a_{t+s}} &= \lambda_{t+s}(1+r) - \lambda_{t+s-1} = 0 \qquad s > 0 \end{aligned}$$

The Euler equation is

$$\frac{\beta[\alpha - (c_{t+s+1} - \gamma c_{t+s})] + \beta^2\gamma[\alpha - (c_{t+s+2} - \gamma c_{t+s+1})]}{[\alpha - (c_{t+s} - \gamma c_{t+s-1})]\beta\gamma[\alpha - (c_{t+s+1} - \gamma c_{t+s})]}(1+r) = 1$$

From the Euler equation it can be shown that consumption evolves according to a third-order difference equation.

Introducing the lag operators $Lc_t = c_{t-1}$ and $L^{-1}c_t = c_{t+1}$, and assuming that $\beta(1+r) \geq 1$, the Euler equation can be written as

$$(1-\gamma L)(1-\frac{1}{\beta(1+r)}L)(1+\beta\gamma L^{-1})c_t = \alpha(1+\beta\gamma)(1-\frac{1}{\beta(1+r)})$$

Hence there are two stable roots $\frac{1}{\gamma}$ and $\beta(1+r)$ and one unstable root $-\frac{1}{\beta\gamma}$. The change in consumption therefore has a saddlepath solution. But as the right-hand side is constant, so is the forward-looking component of the solution. This can be seen by writing the solution as

$$\begin{aligned}(1-\gamma L)(1-\frac{1}{\beta(1+r)}L)c_t &= \alpha(1+\beta\gamma)(1-\frac{1}{\beta(1+r)})\Sigma_{s=0}^{\infty}(-\beta\gamma)^s \\ &= \alpha(1-\frac{1}{\beta(1+r)})\end{aligned}$$

Consequently, the solution reduces to a stable second-order difference equation.

We note that if $\beta(1+r) = 1$ then the solution has a unit root and the right-hand side is zero, i.e. it is

$$\Delta c_{t+1} = \gamma \Delta c_t$$

This solution can also be written as $\Delta(c_{t+1} - \gamma c_t) = 0$, which is the form in which dynamics are introduced in the utility function.

(b) The Lagrangian is now

$$\mathcal{L}_t = \sum_{s=0}^{\infty}\{\beta^s \frac{(c_{t+s} - \gamma c_{t+s-1})^{1-\sigma}}{1-\sigma} + \lambda_{t+s}[x_{t+s} + (1+r)a_{t+s} - c_{t+s} - a_{t+s+1}]\}$$

and the first-order conditions are

$$\begin{aligned}\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} &= \beta^s(c_{t+s} - \gamma c_{t+s-1})^{-\sigma} - \beta^{s+1}\gamma(c_{t+s+1} - \gamma c_{t+s})^{-\sigma} - \lambda_{t+s} = 0 \qquad s \geq 0 \\ \frac{\partial \mathcal{L}_t}{\partial a_{t+s}} &= \lambda_{t+s}(1+r) - \lambda_{t+s-1} = 0 \qquad s > 0\end{aligned}$$

The Euler equation is therefore

$$\frac{\beta(c_{t+1} - \gamma c_t)^{-\sigma} - \beta^2\gamma(c_{t+2} - \gamma c_{t+1})^{-\sigma}}{(c_t - \gamma c_{t-1})^{-\sigma} - \beta\gamma(c_{t+1} - \gamma c_t)^{-\sigma}}(1+r) = 1$$

This is a nonlinear difference equation which has no closed-form solution.

An alternative is to seek a local approximation to the Euler equation which does have a solution. The Euler equation can be re-written in terms of the rate of growth of consumption, η_t say. It can then be shown that the Euler equation can be approximated by a second-order difference equation in η_t.

(c) Comparing these two habit-persistence models with that of Exercise 4.3, we note first that choosing an exogenous level of habit persistence, as in Exercise 4.3, greatly simplifies the analysis. But, especially in empirical finance, it is common to implement exogenous habit persistence by replacing h_t in the Euler equation for Exercise 4.3 by γc_{t-1}. Whilst it it makes sense to base habit persistence on past consumption, it is clear from Exercise 4.4 that this assumption should be introducing from the outset and not after the Euler equation is obtained as the solutions are very different.

4.5. Suppose that households have savings of s_t at the start of the period, consume c_t but have no income. The household budget constraint is $\Delta s_{t+1} = r(s_t - c_t),\ \ 0 < r < 1$ where r is the real interest rate.

(a) If the household's problem is to maximize discounted utility $V_t = \Sigma_{s=0}^{\infty} \beta^s \ln c_{t+s}$where $\beta = \frac{1}{1+r}$,

(i) show that the solution is $c_{t+1} = c_t$

(ii) What is the solution for s_t?

(b) If the household's problem is to maximize expected discounted utility $V_t = E_t \Sigma_{s=0}^{\infty} \beta^s \ln c_{t+s}$

(i) show that the solution is $\frac{1}{c_t} = E_t[\frac{1}{c_{t+1}}]$

(ii) Using a second-order Taylor series expansion about c_t show that the solution can be written as $E_t[\frac{\Delta c_{t+1}}{c_t}] = E_t[(\frac{\Delta c_{t+1}}{c_t})^2]$

(iii) Hence, comment on the differences between the non-stochastic and the stochastic solutions.

Solution

(a) (i) The Lagrangian is

$$\mathcal{L}_t = \sum_{i=0}^{\infty}\{\beta^i \ln c_{t+i} + \lambda_{t+i}[(1+r)s_{t+i} - rc_{t+i} - s_{t+i+1}]\}$$

and the first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial c_{t+i}} &= \beta^i \frac{1}{c_{t+i}} - \lambda_{t+i} r = 0 \qquad i \geq 0 \\
\frac{\partial \mathcal{L}_t}{\partial s_{t+i}} &= \lambda_{t+i}(1+r) - \lambda_{t+i-1} = 0 \qquad i > 0
\end{aligned}$$

The Euler equation is

$$\frac{\beta c_t(1+r)}{c_{t+1}} = 1$$

Hence $c_{t+1} = c_t$.

(ii) From the budget constraint,

$$\begin{aligned}
s_t &= \frac{1}{1+r}(s_{t+1} + rc_t) \\
&= r\Sigma_{i=0}^{\infty}\frac{c_{t+i}}{(1+r)^{i+1}} \\
&= c_t
\end{aligned}$$

(b) (i) Introducing conditional expectations and using stochastic dynamic programming gives the result for the Euler equation given in Chapter 10. This is

$$E_t[\frac{\beta U'(c_{t+1})}{U'(c_t)}(1+r)] = 1$$

As $U(c_t) = \ln c_t$, this becomes

$$\frac{E_t(\frac{1}{c_{t+1}})}{\frac{1}{c_t}}\beta(1+r) = 1$$

hence

$$\frac{1}{c_t} = E_t(\frac{1}{c_{t+1}})$$

(ii) Expanding $E_t(\frac{1}{c_{t+1}})$ in a second-order Taylor series about c_t gives

$$\begin{aligned} E_t(\frac{1}{c_{t+1}}) &\simeq \frac{1}{c_t} - \frac{1}{c_t^2}E_t(c_{t+1} - c_t) + \frac{1}{c_t^3}E_t(c_{t+1} - c_t)^2 \\ &= \frac{1}{c_t} - \frac{1}{c_t}E_t(\frac{\Delta c_{t+1}}{c_t}) + \frac{1}{c_t}E_t(\frac{\Delta c_{t+1}}{c_t})^2 \end{aligned}$$

The Euler equation therefore implies that

$$E_t(\frac{\Delta c_{t+1}}{c_t}) = E_t(\frac{\Delta c_{t+1}}{c_t})^2$$

(iii) The diference between the stochastic and the non-stochastic solutions arises because $E_t(\frac{1}{c_{t+1}}) \neq \frac{1}{E_t(c_{t+1})}$.

4.5. Suppose that households seek to maximize the inter-temporal quadratic objective function

$$V_t = -\frac{1}{2}E_t\sum_{s=0}^{\infty}\beta^s[(c_{t+s} - \gamma)^2 + \phi(a_{t+s+1} - a_{t+s})^2], \quad \beta = \frac{1}{1+r}$$

subject to the budget constraint

$$c_t + a_{t+1} = (1+r)a_t + x_t$$

where c_t is consumption, a_t is the stock of assets and x_t is exogenous.

(a) Comment on the objective function.

(b) Derive expressions for the optimal dynamic behaviors of consumption and the asset stock.

(c) What is the effect on consumption and assets of a permanent shock to x_t of Δx? Comment on the implications for the specification of the utility function.

(d) What is the effect on consumption and assets of a temporary shock to x_t that is unanticipated prior to period t?

(e) What is the effect on consumption and assets of a temporary shock to x_{t+1} that is anticipated in period t?

Solution

(a) The objective function may be regarded as an approximation to the standard problem of maximizing the present value of the discounted utility of consumption where the utility function is approximated by a second-order Taylor series expansion. In the case of x_t stationary about a constant value, the optimal solution to the standard problem is a constant level of consumption which is represented by γ and a constant level of assets a_t. If the aim is constant assets as here, then from the budget constraint

$$c_t = ra_t + x_t$$

hence, intuitively, for stationary x_t

$$E(c_t) = \gamma = rE(a) + E(x_t)$$

(b) The solution may be obtained most easily by eliminating the term in the change in the asset stock in the objective function by substitution from the budget constraint. The objective function then becomes

$$V_t = -\frac{1}{2}E_t \sum_{s=0}^{\infty} \beta^s [(c_{t+s} - \gamma)^2 + \phi(ra_{t+s} + x_{t+s} - c_{t+s})^2]$$

Maximizing this with respect to c_{t+s} and a_{t+s} gives the first-order conditions

$$\begin{aligned}
\frac{\partial V_t}{\partial c_{t+s}} &= -\beta^s E_t[(c_{t+s} - \gamma) - \phi(ra_{t+s} + x_{t+s} - c_{t+s})] = 0, \qquad s \geq 0 \\
\frac{\partial V_t}{\partial a_{t+s}} &= -\phi\beta^s r E_t(ra_{t+s} + x_{t+s} - c_{t+s}) = 0, \qquad s > 0
\end{aligned}$$

Hence

$$E_t a_{t+s} = \frac{1}{r}E_t(c_{t+s} - x_{t+s})$$

and so

$$\begin{aligned}
E_t c_{t+s} &= \gamma, \qquad s \geq 0 \\
E_t a_{t+s} &= \frac{1}{r}(\gamma - E_t x_{t+s}), \qquad s > 0
\end{aligned}$$

where a_t is given. In long-run static equilibrium this implies that $E(c) = E(x) = \gamma$ and $E(a) = 0$. If x_t were growing over time then intuitively the optimal level of consumption would grow with x_t.

(c) A permanent increase in x_t unanticipated prior to period t will have no effect on c_t which will remain equal to γ. But it will reduce the stock of assets by $\frac{\Delta x}{r}$, where Δx is the permanent change in x_t.

This is an implausible result. It is due to the constant target of γ for consumption which, as noted, only makes sense if the mean of x_t stays constant. A more plausible specification for the utility function would set target consumption at the new mean value of x_t, namely, $E(x_t)+\Delta x$. In this case optimal long-run consumption would increase by Δx and assets would remain unchanged.

(d) An unanticipated temporary increase in x_t of Δx_t will have no effect on c_t but will raise a_{t+1}, a_{t+2}, a_{t+3},..... From the budget constraint

$$\begin{aligned}
a_{t+1} &= (1+r)a_t + x_t + \Delta x_t - \gamma \\
a_{t+2} &= (1+r)a_{t+1} + x_t - \gamma \\
&= (1+r)^2 a_t + (2+r)(x_t - \gamma) + (1+r)\Delta x_t \\
a_{t+3} &= (1+r)a_{t+1} + x_t - \gamma \\
&= (1+r)^3 a_t + \Sigma_{i=0}^{3}(1+r)^{i-1}(x_t - \gamma) + (1+r)^2 \Delta x_t
\end{aligned}$$

Hence, $\lim\limits_{n\to\infty} E_t a_{t+n} = \infty$. Again this makes little sense.

Recalling our previous discussion that a plausible target for consumption is $E(x_t)$ rather than γ, the second term for a_{t+3} would be zero. It then follows from the budget constraint that $a_{t+1} = (1+r)a_t$ which requires that $a_t = 0$, i.e. consumption is entirely from non-asset income and total savings would be zero. The first term for a_{t+3} is now eliminated. Finally, we note that shocks to x_t are a mean zero process. This implies that in the future negative shocks are expected to offset positive shocks implying that on average a_t is zero.

(e) A temporary increase in x_{t+1} anticipated in period t does not affect consumption as $E_t c_{t+s} = \gamma$ for all s. But it will affect the stock of assets in every period from period $t+1$. In period $t+1$

$$\begin{aligned} E_t a_{t+1} &= \frac{1}{r}(\gamma - E_t x_{t+1}) \\ &= \frac{1}{r}(\gamma - x_t - \Delta x) \end{aligned}$$

As $E_t x_{t+1} = x_t$, in period $t+2$, from the budget constraint,

$$\begin{aligned} E_t a_{t+2} &= (1+r)E_t a_{t+1} + E_t x_{t+2} - \gamma \\ &= \frac{1}{r}(\gamma - x_t) - \frac{1+r}{r}\Delta x \end{aligned}$$

Similarly, in period $t+3$

$$\begin{aligned} E_t a_{t+3} &= (1+r)E_t a_{t+2} + E_t x_{t+3} - \gamma \\ &= \frac{1}{r}(\gamma - x_t) - \frac{(1+r)^2}{r}\Delta x \end{aligned}$$

Following the logic of parts (c) and (d), $\gamma = x_t$, $a_t = 0$ and $E(\Delta x) = 0$, hence $\lim_{n\to\infty} E_t a_{t+n} = 0$.

4.6. Households live for periods t and $t+1$. The discount factor for period $t+1$ is $\beta = 1$. They receive exogenous income x_t and x_{t+1}, where the conditional distribution of income in period $t+1$ is $N(x_t, \sigma^2)$, but they have no assets.

(a) Find the level of c_t that maximises $V_t = U(c_t) + E_t U(c_{t+1})$ if the utility function is quadratic: $U(c_t) = -\frac{1}{2}c_t^2 + \alpha c_t$, $(\alpha > 0)$.

(b) Calculate the conditional variance of this level of c_t and hence comment on what this implies about consumption smoothing.

Solution

(a) This problem can be expressed as maximizing

$$V_t = -\frac{1}{2}(c_t^2 + E_t c_{t+1}^2) + \alpha(c_t + E_t c_{t+1})\}$$

with respect to c_t and c_{t+1} subject the two-period inter-temporal constraint

$$c_t + E_t c_{t+1} = x_t + E_t x_{t+1}$$

The Lagrangian is

$$\mathcal{L}_t = -\frac{1}{2}(c_t^2 + E_t c_{t+1}^2) + \alpha(c_t + E_t c_{t+1}) + \lambda(x_t + E_t x_{t+1} - c_t - E_t c_{t+1})$$

The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial c_t} &= -c_t + \alpha - \lambda = 0 \\
\frac{\partial \mathcal{L}_t}{\partial c_{t+1}} &= -E_t c_{t+1} + \alpha - \lambda = 0
\end{aligned}$$

Hence,$c_t = E_t c_{t+1}$. From the two-period budget constraint,

$$c_t = \frac{1}{2}(x_t + E_t x_{t+1}) = x_t$$

Optimal consumption in period t is therefore half cumulated total expected income which is x_t.

(b) As x_t as known in period t, the conditional distribution of c_t is $N(x_t, \frac{\sigma^2}{4})$. Hence inter-temporal optimization has resulted in reducing the volatility of consumption compared with that of income. In other words, consumption smoothing has occurred despite the absence of assets.

4.7. An alternative way of treating uncertainty is through the use of contingent states s^t, which denotes the state of the economy up to and including time t, where $s^t = (s_t, s_{t-1}, ...)$ and there are S different possible states with probabilities $p(s^t)$. The aim of the household can then be expressed as maximizing over time and over all possible states of nature the expected discounted sum of current and future utility

$$\Sigma_{t,s} \beta^t p(s^t) U[c(s^t)]$$

subject to the budget constraint in state s^t

$$c(s^t) + a(s^t) = [1 + r(s^t)]a(s^{t-1}) + x(s^t)$$

where $c(s^t)$ is consumption, $a(s^t)$ are assets and $x(s^t)$ is exogenous income in state s^t. Derive the optimal solutions for consumption and assets.

Solution

The Lagrangian for this problem is

$$\mathcal{L} = \Sigma_{t,s}\{\beta^t p(s^t)U[c(s^t)] + \lambda(s^t)[(1 + r(s^t))a(s^{t-1}) + x(s^t) - c(s^t) - a(s^t)]\}$$

where $\lambda(s^t)$ is the Lagrange multiplier in state s^t. The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c(s^t)} &= \beta^t p(s^t)U'[c(s^t)] - \lambda(s^t) = 0 \\
\frac{\partial \mathcal{L}}{\partial a(s^t)} &= \lambda(s^{t+1})[1 + r(s^{t+1})] - \lambda(s^t) = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda(s^t)} &= [1 + r(s^t)]a(s^{t-1}) + x(s^t) - c(s^t) - a(s^t) = 0
\end{aligned}$$

Hence, eliminating the Lagrange multipliers, gives the Euler equation

$$\frac{\beta p(s^{t+1})U'[c(s^{t+1})]}{p(s^t)U'[c(s^t)]}[1 + r(s^{t+1})] = 1$$

We note that

$$\frac{p(s^{t+1})}{p(s^t)} = \frac{p(s_{t+1}, s^t)}{p(s^t)} = p(s^{t+1}/s^t) = p(s_{t+1}/s^t)$$

i.e. the conditional probability of s_{t+1} given s^t. Consequently, the right-hand side of the Euler equation is, once more, the conditional expectation given information at time t of the discounted value of one plus the rate of return evaluated in terms of period t utility. The optimal solutions for consumption and assets are therefore as before.

4.8. Suppose that firms face additional (quadratic) costs associated with the accumulation of capital and labor so that firm profits are

$$\Pi_t = Ak_t^\alpha n_t^{1-\alpha} - w_t n_t - i_t - \frac{1}{2}\mu(\Delta k_{t+1})^2 - \frac{1}{2}\nu(\Delta n_{t+1})^2$$

where $\mu, \nu > 0$, the real wage w_t is exogenous and $\Delta k_{t+1} = i_t - \delta k_t$. If firms maximize the expected present value of the firm $E_t[\Sigma_{s=0}^{\infty}(1 + r)^{-s}\Pi_{t+s}]$,

(a) derive the demand functions for capital and labor in the long run and the short run.

(b) What would be the response of capital and labor demand to

(i) a temporary increase in the real wage in period t, and

(ii) a permanent increase in the real wage from period t?

Solution

Combining the given information and eliminating i_t, the firm's problem is to maximize

$$V_t = E_t \Sigma_{s=0}^{\infty}(1+r)^{-s}[Ak_{t+s}^{\alpha}n_{t+s}^{1-\alpha} - w_{t+s}n_{t+s} - k_{t+s+1} + (1-\delta)k_{t+s} - \frac{1}{2}\mu(\Delta k_{t+s+1})^2 - \frac{1}{2}\nu(\Delta n_{t+s})^2]$$

The first-order conditions are

$$\begin{aligned}
\frac{\partial V_t}{\partial n_{t+s}} &= E_t\{(1+r)^{-s}[(1-\alpha)Ak_{t+s}^{\alpha}n_{t+s}^{-\alpha} - w_{t+s} - \nu\Delta n_{t+s}] + (1+r)^{-(s+1)}\nu\Delta n_{t+s+1}]\} = 0, \quad s \geq 0 \\
\frac{\partial V_t}{\partial k_{t+s}} &= E_t\{(1+r)^{-s}[\alpha Ak_{t+s}^{\alpha-1}n_{t+s}^{1-\alpha} + 1 - \delta + \mu\Delta k_{t+s+1}] - (1+r)^{-(s-1)}[1 - \mu\Delta k_{t+s}]\} = 0, \quad s > 0
\end{aligned}$$

For $s = 0$ the labor first-order condition can be written as

$$\Delta n_t = \frac{1}{1+r}\Delta n_{t+1} + \frac{1}{\nu}[(1-\alpha)Ak_t^{\alpha}n_t^{-\alpha} - w_t] \tag{5}$$

Thus, in steady-state, labour is paid its marginal product

$$F_n' = (1-\alpha)A\left(\frac{k}{n}\right)^{\alpha} = w$$

For $s = 1$ the first-order condition for capital can be written as

$$\Delta k_{t+1} = \frac{1}{1+r}\Delta k_{t+2} + \frac{1}{\mu(1+r)}[\alpha Ak_{t+1}^{\alpha-1}n_{t+1}^{1-\alpha} - \delta + r] \tag{6}$$

implying that in steady state capital is paid its net marginal product

$$F_k' - \delta = \alpha A\left(\frac{k}{n}\right)^{-(1-\alpha)} - \delta = r$$

Given r and w we can solve for the steady-state values of k and n from the two long-run conditions.

In the short run, the dynamic behavior of labor and capital is obtained from equations (5) and (6). Both labor and capital have forward-looking solutions:

$$\begin{aligned}
\Delta n_t &= \frac{1}{\nu} E_t \Sigma_{s=0}^{\infty} (1+r)^{-s} [(1-\alpha) A k_{t+s}^{\alpha} n_{t+s}^{-\alpha} - w_{t+s}] \\
\Delta k_{t+1} &= \frac{1}{\mu} E_t \Sigma_{s=0}^{\infty} (1+r)^{-(s+1)} [\alpha A k_{t+s+1}^{\alpha-1} n_{t+s+1}^{1-\alpha} - \delta + r]
\end{aligned}$$

Thus current changes in labor and capital respond instantly to the discounted sum of expected future departures of their marginal products from their long-run values. And since the marginal products depend on both labor and capital, the two current changes are determined simultaneously. We note that, despite the costs of adjustment of labor and capital, the changes are instantaneous.

Chapter 5

5.1. In an economy that seeks to maximize $\Sigma_{s=0}^{\infty}\beta^s \ln c_{t+s}$ ($\beta = \frac{1}{1+r}$) and takes output as given the government finances its expenditures by taxes on consumption at the rate τ_t and by debt.

(a) Find the optimal solution for consumption given government expendtitures, tax rates and government debt.

(b) Starting from a position where the budget is balanced and there is no government debt, analyse the consequences of

(i) a temporary increase in government expenditures in period t,

(ii) a permanent increase in government expenditures from period t.

Solution

(a) The national income identity for this economy can be written as

$$y_t = c_t + g_t$$

and the government budget constraint is

$$\begin{aligned}\Delta b_{t+1} + T_t &= g_t + rb_t \\ T_t &= \tau_t c_t\end{aligned}$$

The resource constraint for the economy is therefore

$$y_t = (1+\tau_t)c_t + rb_t - \Delta b_{t+1}$$

The Lagrangian is

$$\mathcal{L}_t = \sum\nolimits_{s=0}^{\infty} \{\beta^s \ln c_{t+s} + \lambda_{t+s}[y_{t+s} - (1+\tau_{t+s})c_{t+s} - (1+r)b_{t+s} + b_{t+s+1}]\}$$

The first-order conditions are

$$\begin{aligned}\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} &= \beta^s \frac{1}{c_{t+s}} - \lambda_{t+s}(1+\tau_{t+s}) = 0 \qquad\qquad s \geq 0 \\ \frac{\partial \mathcal{L}_t}{\partial b_{t+s}} &= \lambda_{t+s}(1+r) - \lambda_{t+s-1} = 0 \qquad\qquad s > 0\end{aligned}$$

The Euler equation is therefore

$$\frac{\beta c_t}{c_{t+1}} \frac{(1+\tau_t)}{(1+\tau_{t+1})}(1+r) = 1$$

or, as $\beta = \frac{1}{1+r}$,

$$\frac{c_t}{c_{t+1}} \frac{(1+\tau_t)}{(1+\tau_{t+1})} = 1$$

In steady-state, when consumption is constant, so are the consumption tax rates. But in the short run, in general, tax rates must vary in order to satisfy the government budget constraint and this will affect consumption.

(b) From the government budget constraint we have

$$\begin{aligned} b_t &= \frac{1}{1+r}[b_{t+1} + \tau_t c_t - g_t] \\ &= \sum\nolimits_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^{s+1} (\tau_{t+s} c_{t+s} - g_{t+s}) \end{aligned}$$

if the transversality condition $\lim_{s\to\infty} \Sigma_{s=0}^{\infty}(1+r)^{-(s+1)} b_{t+s} = 0$ holds. If the initial level of debt is zero and the budget is balanced then

$$0 = \sum\nolimits_{s=1}^{\infty} \left(\frac{1}{1+r}\right)^{s+1} (\tau_{t+s} c_{t+s} - g_{t+s})$$

as $g_t = \tau_t c_t$. From the Euler equation

$$(1+\tau_t)c_t = (1+\tau_{t+1})c_{t+1}$$

Hence,

$$\begin{aligned} 0 &= \sum\nolimits_{s=1}^{\infty} \left(\frac{1}{1+r}\right)^{s+1} [(1+\tau_{t+s})c_{t+s} - c_{t+s} - g_{t+s}] \\ &= \frac{(1+\tau_t)c_t}{r} - \sum\nolimits_{s=1}^{\infty} \left(\frac{1}{1+r}\right)^{s+1} (c_{t+s} + g_{t+s}) \end{aligned}$$

or, as $g_t = \tau_t c_t$,

$$c_t + g_t = r \sum\nolimits_{s=1}^{\infty} \left(\frac{1}{1+r}\right)^{s+1} (c_{t+s} + g_{t+s})$$

And, as $y_t = c_t + g_t$, with y_t given, it follows that for all $s \geq 0$ an increase in g_{t+s} must be match in each period by an equal reduction in c_{t+s}. But this implies that tax revenues would be cut if tax rates were unchanged. Because the government budget constraint must be satisfied, it follows that either τ_{t+s}, or debt, or both must increase as $\tau_{t+s} \leq 1$.

(i) A temporary increase in g_t of Δg is offset by an equal temporary fall in c_t. Depending on the size of Δg, this could be paid for by an increase in τ_t of $\Delta\tau$ if

$$g_t + \Delta g = (\tau_t + \Delta\tau)(c_t - \Delta g)$$

and

$$\tau_t + \Delta\tau \leq 1$$

But debt finance of

$$\Delta b = g_t + \Delta g - (\tau_t + \Delta\tau)(c_t - \Delta g)$$

would be required if $\tau_t + \Delta\tau$ would otherwise exceed unity.

(ii) The general principle established in Chapter 5 that, in order to satisfy the intertemporal budget constraint, a permanent increase in government expenditures must be paid for by additional taxation, puts a constraint on the allowable size of the permanent increase in government expenditures. Assuming a constant level of output and a constant tax rate of τ_t in each period, the maximum possible increase in taxes is $1 - \tau_t$ and hence the maximum increase in government expenditures must satisfy

$$g_t + \Delta g \leq c_t - \Delta g$$

And as initially $g_t = \tau_t c_t$ we obtain

$$\Delta g \leq \frac{1-\tau}{2\tau} g_t$$

or

$$\frac{\Delta g}{g_t} \leq \frac{1}{2} \frac{\Delta\tau}{\tau_t}$$

Hence, the maximum proportional permanent rate of change of government expenditures cannot exceed half the proportional rate of increase in the rate of tax.

5.2. Suppose that government expenditures g_t are all capital expenditures and the stock of government capital G_t is a factor of production. If the economy is described by

$$\begin{aligned} y_t &= c_t + i_t + g_t \\ y_t &= Ak_t^{\alpha} G_t^{1-a} \\ \Delta k_{t+1} &= i_t - \delta k_t \\ \Delta G_{t+1} &= g_t - \delta G_t \end{aligned}$$

and the aim is to maximize $\Sigma_{s=0}^{\infty} \beta^s \ln c_{t+s}$,

(a) obtain the optimal solution.

(b) Comment on how government expenditures are being implicitly paid for in this problem.

Solution

(a) The resource constraint is

$$Ak_t^{\alpha} G_t^{1-a} = c_t + k_{t+1} - (1-\delta)k_t + G_{t+1} - (1-\delta)G_t$$

The Lagrangian for this problem is therefore

$$\mathcal{L}_t = \Sigma_{s=0}^{\infty}\{\beta^s \ln c_{t+s} + \lambda_{t+s}[Ak_{t+s}^{\alpha} G_{t+s}^{1-a} - c_{t+s} - k_{t+s+1} + (1-\delta)k_{t+s} - G_{t+s+1} + (1-\delta)G_{t+s}\}$$

The first-order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}_t}{\partial c_{t+s}} &= \beta^s \frac{1}{c_{t+s}} - \lambda_{t+s} = 0, \qquad s \geq 0 \\ \frac{\partial \mathcal{L}_t}{\partial k_{t+s}} &= \lambda_{t+s}[\alpha A k_{t+s}^{\alpha-1} G_{t+s}^{1-a} + 1 - \delta] - \lambda_{t+s-1} = 0, \qquad s > 0 \\ \frac{\partial \mathcal{L}_t}{\partial G_{t+s}} &= \lambda_{t+s}[(1-\alpha) A k_{t+s}^{\alpha} G_{t+s}^{-a} + 1 - \delta] - \lambda_{t+s-1} = 0, \qquad s > 0 \end{aligned}$$

The consumption Euler equation for $s = 1$ is therefore

$$\frac{\beta c_t}{c_{t+1}}[\alpha A k_{t+1}^{\alpha-1} G_{t+1}^{1-a} + 1 - \delta] = \frac{\beta c_{t+1}}{c_t}[(1-\alpha) A k_{t+1}^{\alpha} G_{t+1}^{-a} + 1 - \delta] = 1$$

Suppose that in steady state consumption grows at the rate η, and the two types of capital have the same rate of return r_{t+1} where

$$r_{t+1} = \alpha A k_{t+1}^{\alpha-1} G_{t+1}^{1-a} - \delta = (1-\alpha) A k_{t+1}^{\alpha} G_{t+1}^{-a} - \delta$$

In steady state, therefore,

$$\frac{k}{G} = \frac{\alpha}{1-\alpha}$$

and the steady-state rate of growth of consumption is

$$\eta = r - \theta$$

Consequently, if $\beta = \frac{1}{1+\theta}$, then growth is zero: private saving is continued until its rate of return is the rate of return to private capital. Government capital expenditures, if optimal, continue until their rate of return is the same as the rate of return on private capital.

(b) Government capital expenditures must be paid for by taxes or borrowing. As there is no debt in the model we may assume that taxes are being used instead and, in particular, lump-sum taxes. To make this explicit we could re-write the national income identity so that private saving (income after taxes less total expenditures) equals the government deficit:

$$(y_t - T_t) - c_t - i_t = g_t - T_t$$

5.3. Suppose that government finances its expenditures through lump-sum taxes T_t and debt b_t but there is a cost of collecting taxes given by

$$\Phi(T_t) = \phi_1 T_t + \frac{1}{2}\phi_2 T_t^2, \qquad \Phi'(T_t) \geq 0$$

If the national income identity and the government budget constraint are

$$\begin{aligned} y_t &= c_t + g_t + \Phi(T_t) \\ \Delta b_{t+1} + T_t &= g_t + rb_t + \Phi(T_t) \end{aligned}$$

where output y_t and government expenditures g_t are exogenous, and the aim is to maximize $\Sigma_{s=0}^{\infty}\beta^s U(c_{t+s})$ for $\beta = \frac{1}{1+r}$,

(a) find the optimal solution for taxes.

(b) What is the household budget constraint?

(c) Analyse the effects on taxes, debt and consumption of

(i) a temporary increase government expenditures in period t

(ii) an increase in output.

Solution

(a) Taking into account the two constraints, as both y_t and g_t are exogenous, the Lagrangian is

$$\begin{aligned} \mathcal{L}_t &= \Sigma_{s=0}^{\infty}\{\beta^s U(c_{t+s}) + \lambda_{t+s}[y_{t+s} - c_{t+s} - g_{t+s} - \phi_1 T_{t+s} - \frac{1}{2}\phi_2 T_{t+s}^2] \\ &\quad + \mu_{t+s}[g_{t+s} - b_{t+s+1} - (1+r)b_{t+s} - T_{t_{t+s}} + \phi_1 T_{t+s} + \frac{1}{2}\phi_2 T_{t+s}^2]\} \end{aligned}$$

The first-order conditions with respect to consumption, taxes and debt are

$$\begin{aligned} \frac{\partial \mathcal{L}_t}{\partial c_{t+s}} &= \beta^s U'(c_{t+s}) - \lambda_{t+s} = 0, \qquad s \geq 0 \\ \frac{\partial \mathcal{L}_t}{\partial T_{t+s}} &= -(\phi_1 + \phi_2 T_{t+s})\lambda_{t+s} - \mu_{t+s}(1 - \phi_1 - \phi_2 T_{t+s}) = 0, \qquad s \geq 0 \\ \frac{\partial \mathcal{L}_t}{\partial b_{t+s}} &= (1+r)\mu_{t+s} - \mu_{t+s-1} = 0, \qquad s > 0 \end{aligned}$$

It follows that the consumption Euler equation for $s = 1$ is

$$\frac{\beta U'(c_{t+1})}{U'(c_t)} \frac{\phi_1 + \phi_2 T_{t+1}}{\phi_1 + \phi_2 T_t} \frac{1 - \phi_1 - \phi_2 T_t}{1 - \phi_1 - \phi_2 T_{t+1}}(1+r) = 1$$

Recalling that $\beta(1+r) = 1$, there are two solutions to the Euler equation:

(i) $c_t = c_{t+1}$ and $T_t = T_{t+1}$, implying that optimal consumption and taxes are constant. This solution is a general equilibrium version of the partial equilibrium solution of Chapter 5 (see 5.7.2.5).

(ii) The alternative solution is the dynamic equation

$$\frac{U'(c_t)}{1 - \frac{1}{\phi_1 + \phi_2 T_t}} = \frac{U'(c_{t+1})}{1 - \frac{1}{\phi_1 + \phi_2 T_{t+1}}}$$

(b) The implied household budget constraint is obtained by combining the national income identity and the government budget constraint and is

$$y_t - T_t + rb_t = c_t + \Delta b_{t+1}$$

(c) We take a constant steady state as the optimal solution in answering part (c) and we re-write the optimal solutions introducing conditional expectations so that $c_t = E_t c_{t+1}$ and $T_t = E_t T_{t+1}$.

(i) Assume that $g_t = g + \varepsilon_t$, where ε_t is the temporary increase in period t. From the government budget constraint

$$\begin{aligned} b_t &= \frac{T_t - g_t}{1+r} + \frac{1}{1+r} E_t b_{t+1} \\ &= E_t \sum\nolimits_{s=0}^{\infty} \left(\frac{T_{t+s} - g_{t+s}}{(1+r)^{s+1}} \right) \\ &= \frac{T_t}{r} - E_t \sum\nolimits_{s=0}^{\infty} \frac{g_{t+s}}{(1+r)^{s+1}} \\ &= \frac{T_t}{r} - \frac{g}{r} - \frac{\varepsilon_t}{1+r} \end{aligned}$$

As b_t is given, it follows that in period t

$$T_t = g + rb_t + \frac{r}{1+r}\varepsilon_t$$

T_t must therefore increase by $\frac{r}{1+r}\varepsilon_t$ in order that the GBC is satisfied. And in period $t+1$, as $E_t \varepsilon_{t+1} = 0$ and $E_t T_{t+1} = T_t$, it follows that

$$b_{t+1} = \frac{T_{t+1}}{r} - \frac{g}{r} - \frac{\varepsilon_{t+1}}{1+r}$$

with

$$\begin{aligned}
E_t b_{t+1} &= \frac{E_t T_{t+1}}{r} - \frac{g}{r} \\
&= \frac{T_t}{r} - \frac{g}{r} \\
&= b_t + \frac{\varepsilon_t}{1+r}
\end{aligned}$$

From the household budget constraint

$$\begin{aligned}
c_t &= y_t - T_t + rb_t - \Delta b_{t+1} \\
&= y_t - (g + rb_t + \frac{r}{1+r}\varepsilon_t) + (1+r)b_t - (b_t + \frac{\varepsilon_t}{1+r}) \\
&= y_t - g - \varepsilon_t
\end{aligned}$$

Thus consumption is cut by ε_t. This result for consumption could also be obtained directly from the national income identity.

(ii) An increase in income, whether temporary or permanent, will raise consumption and have no effect on either taxes or debt.

5.4. Assuming that output growth is zero, inflation and the rate of growth of the money supply are π, that government expenditures on goods and services plus transfers less total taxes equals z and the real interest rate is $r > 0$,

(a) what is the minimum rate of inflation consistent with the sustainability of the fiscal stance in an economy that has government debt?

(b) How do larger government expenditures affect this?

(c) What are the implications for reducing inflation?

Solution

(a) We begin by stating the government budget constraint in terms of proportions of GDP (see Chapter 5, section 5.4)

$$\frac{P_t g_t}{P_t y_t} + \frac{P_t h_t}{P_t y_t} + \frac{(1+R_t)B_t}{P_t y_t} = \frac{P_t T_t}{P_t y_t} + \frac{B_{t+1}}{P_t y_t} + \frac{M_{t+1}}{P_t y_t} - \frac{M_t}{P_t y_t}$$

When growth γ_t is zero, inflation π_t is constant and money grows at the rate π this can be written as

$$\frac{g_t}{y_t} + \frac{h_t}{y_t} + (1+R_t)\frac{b_t}{y_t} = \frac{T_t}{y_t} + (1+\pi)\frac{b_{t+1}}{y_{t+1}} + \pi\frac{m_t}{y_t}$$

where $\frac{m_t}{y_t} = \frac{m}{y}$, a constant. In terms of the ratio of the primary deficit to GDP $\frac{d_t}{y_t}$ this becomes

$$\begin{aligned}
\frac{d_t}{y_t} &= \frac{g_t}{y_t} + \frac{h_t}{y_t} - \frac{T_t}{y_t} - \pi\frac{m_t}{y_t} \\
&= \frac{z - \pi m}{y} \\
&= -(1+R_t)\frac{b_t}{y_t} + (1+\pi)\frac{b_{t+1}}{y_{t+1}}
\end{aligned}$$

As $R > \pi$, we solve the government budget constraint *forwards* to obtain the sustainability conditions

$$\begin{aligned}
\frac{b_t}{y_t} &\leq \frac{1}{1+R}\sum_{s=0}^{\infty}\left(\frac{1+\pi}{1+R}\right)^s \left(\frac{-d_{t+s}}{y_{t+s}}\right) \\
&\leq \frac{1}{R-\pi}\frac{d}{y} = \frac{1}{r}\frac{\pi m - z}{y} \\
\lim_{n\to\infty}\left(\frac{1+\pi}{1+R}\right)^n \frac{b_{t+n}}{y_{t+n}} &= 0
\end{aligned}$$

It follows that, given these conditions, the minimum rate of inflation consistent with fiscal sustainability is

$$\pi \geq \frac{z + rb}{m}$$

This is to ensure that seigniorage revenues are sufficient to pay for government expenditures net of taxes.

(b) If government expenditures increase then inflation must be allowed to rise too in order that seigniorage revenues increase to pay for the additional expenditures.

(c) Conversely, in order to reduce inflation, either government expenditures must be reduced or tax revenues increased.

5.5. Consider an economy without capital that has proportional taxes on consumption and labor and is described by the following equations

$$
\begin{aligned}
y_t &= An_t^{\alpha} = c_t + g_t \\
g_t + rb_t &= \tau_t^c c_t + \tau_t^w w_t n_t + \Delta b_{t+1} \\
U(c_t, l_t) &= \ln c_t + \gamma \ln l_t \\
1 &= n_t + l_t
\end{aligned}
$$

(a) State the household budget constraint.

(b) If the economy seeks to maximize $\Sigma_{s=0}^{\infty} \beta^s U(c_{t+s}, l_{t+s})$, where $\beta = \frac{1}{1+r}$, derive the optimal steady-state levels of consumption and employment for given g_t, b_t and tax rates.

Solution

(a) The household budget constraint can be obtained from the national income identity and the government budget constraint by eliminating g_t. It can be written as

$$y_t + rb_t = (1 + \tau_t^c)c_t + \tau_t^w w_t n_t + \Delta b_{t+1}$$

Or, as household labor income satisfies $w_t n_t = y_t$,

$$(1 - \tau_t^w) w_t n_t + rb_t = (1 + \tau_t^c)c_t + \Delta b_{t+1}$$

(b) The problem is to maximize inter-temporal utility subject to the national income constraint and the government budget constraint. The Lagrangian for this problem can be written as

$$\mathcal{L}_t = \sum\nolimits_{s=0}^{\infty} \{\beta^s(\ln c_{t+s} + \gamma \ln l_{t+s}) + \lambda_{t+s}[An_{t+s}^{\alpha} - (1+\tau_{t+s}^c)c_{t+s} - \tau_{t+s}^w w_{t+s} n_{t+s} + (1+r)b_{t+s} - b_{t+s+1}]\}$$

The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} &= \beta^s \frac{1}{c_{t+s}} - \lambda_{t+s}(1+\tau^c_{t+s}) = 0 \qquad\qquad s \geq 0 \\
\frac{\partial \mathcal{L}_t}{\partial n_{t+s}} &= -\beta^s \frac{\gamma}{1-n_{t+s}} + \lambda_{t+s}(\alpha A n^{\alpha-1}_{t+s} - \tau^w_{t+s} w_{t+s}) = 0 \qquad\qquad s \geq 0 \\
\frac{\partial \mathcal{L}_t}{\partial b_{t+s}} &= \lambda_{t+s}(1+r) - \lambda_{t+s-1} = 0 \qquad\qquad s > 0
\end{aligned}$$

The consumption Euler equation is therefore

$$\frac{\beta c_t}{c_{t+1}} \frac{(1+\tau^c_t)}{(1+\tau^c_{t+1})}(1+r) = 1$$

Consequently, as $\beta = \frac{1}{1+r}$, with constant consumption tax rates, consumption is also constant.

From the first-order conditions for consumption and work we obtain the steady-state solution

$$\frac{\gamma c}{(1-n)} = \frac{\alpha A n^{\alpha-1} - \tau^w w}{1+\tau^c}$$

If labor is paid its marginal product then $w_t = \alpha A n_t^{\alpha-1}$, and so

$$\frac{\gamma c}{(1-n)} = \frac{(1-\tau^w)w}{1+\tau^c}$$

This can be shown to be the same solution that would have been obtained by maximizing $\Sigma_{s=0}^{\infty}\beta^s U(c_{t+s}, l_{t+s})$ subject to the household budget constraint. Solving this together with the household budget constraint gives the solutions for c, l and n. Thus

$$\begin{aligned}
c_t &= \frac{(1-\tau^w)w + rb}{(1+\gamma)(1+\tau^c)} \\
n_t &= \frac{(1-\tau^w)w - \gamma rb}{(1+\gamma)(1-\tau^w)w}
\end{aligned}$$

5.6 (a) What is the Ramsey problem of optimal taxation?

(b) For Exercise 5 find the optimal rates of consumption and labor taxes by solving the associated Ramsey problem.

Solution

(a) The Ramsey problem is for government to find the optimal tax rates that are consistent with the optimality conditions of households who are taking tax rates as given.

(b) From the first-order conditions in the solution to Exercise 5 we have $\frac{\lambda_{t-1}}{\lambda_t} = 1 + r$. The constraint imposed by households on government decisions which take into account the optimality conditions of households can therefore be written as

$$(1 + \tau_t^c)c_t + b_{t+1} = (1 - \tau_t^w)w_t n_t + \frac{\lambda_{t-1}}{\lambda_t} b_t$$

This can be solved forwards to give the inter-temporal constraint

$$\lambda_{t-1}b_t = \sum\nolimits_{s=0}^{\infty} \lambda_{t+s}[(1 + \tau_{t+s}^c)c_{t+s} - (1 - \tau_{t+s}^w)w_{t+s}n_{t+s}]$$

provided the transversality condition $\lim_{n\to\infty} \lambda_{t+n}b_{t+n+1} = 0$ holds. Using the first-order conditions for consumption and work, and recalling that labor is paid its marginal product, it can be re-written as the implementability condition

$$\begin{aligned}
\lambda_{t-1}b_t &= \sum\nolimits_{s=0}^{\infty} \beta^s (U_{c,t+s}c_{t+s} - U_{l,t+s}n_{t+s}) \\
&= \sum\nolimits_{s=0}^{\infty} \beta^s (\frac{1}{c_{t+s}} c_{t+s} + \frac{\gamma}{1 - n_{t+s}} n_{t+s}) \\
&= \sum\nolimits_{s=0}^{\infty} \beta^s \frac{1 - (1 - \gamma)n_{t+s}}{1 - n_{t+s}}
\end{aligned}$$

We note that the left-hand side is pre-determined at time t.

The government's problem is to maximize the inter-temporal utility of households subject to the implementability condition and the economy's resource constraint. The Lagrangian can be written as

$$\begin{aligned}
\mathcal{L}_t &= \sum\nolimits_{s=0}^{\infty} \{\beta^s(\ln c_{t+s} + \gamma \ln l_{t+s}) + \phi_{t+s}[w_{t+s}n_{t+s} - c_{t+s} - g_{t+s}]\} \\
&\quad +\mu[\sum\nolimits_{s=0}^{\infty} \beta^s \frac{\gamma}{1 - n_{t+s}} - \lambda_{t-1}b_t]
\end{aligned}$$

Defining

$$V(c_{t+s}, l_{t+s}, \mu) = \ln c_{t+s} + \gamma \ln l_{t+s} + \mu \frac{\gamma}{1 - n_{t+s}}$$

The Lagrangian is then

$$\begin{aligned}\mathcal{L}_t &= \sum\nolimits_{s=0}^{\infty}\{\beta^s V(c_{t+s}, l_{t+s}, \mu) \\ &\quad +\phi_{t+s}[w_{t+s}n_{t+s} - c_{t+s} - g_{t+s}]\} - \mu\lambda_{t-1}b_t\end{aligned}$$

The first-order conditions for consumption and labor are

$$\begin{aligned}\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} &= \beta^s V_{c,t+s} - \phi_{t+s} = 0 \qquad s \geq 0 \\ \frac{\partial \mathcal{L}_t}{\partial n_{t+s}} &= -\beta^s V_{l,t+s} + \phi_{t+s}w_{t+s} = 0 \qquad s \geq 0\end{aligned}$$

We now consider the implications of these conditions for the optimal choice of the three tax rates.

From the first-order conditions and the marginal condition for labor

$$\begin{aligned}\frac{V_{l,t}}{V_{c,t}} &= w_t \\ &= \frac{(1+\mu)U_{l,t} + \mu(U_{cl,t}c_t - U_{ll,t}n_t)}{(1+\mu)U_{c,t} + \mu(U_{cc,t}c_t - U_{lc,t}l_t)} \\ &= \frac{1+\tau_t^c}{1-\tau_t^w}\frac{U_{l,t}}{U_{c,t}} \\ &= -\frac{1+\tau_t^c}{1-\tau_t^w}\frac{\gamma c_t}{1-n_t}\end{aligned}$$

As the utility function is homothetic, i.e. for any θ

$$\frac{U_c[\theta c, \theta l)]}{U_l[\theta c, \theta l)]} = \frac{U_c(c, l)}{U_l(c, l)}$$

differentiating with respect to θ and evaluating at $\theta = 1$ gives

$$\frac{U_{cc,t}c_t - U_{lc,t}l_t}{U_{c,t}} = \frac{U_{cl,t}c_t - U_{ll,t}l_t}{U_{l,t}}$$

Thus

$$\frac{V_{l,t}}{V_{c,t}} = \frac{U_{l,t}}{U_{c,t}}$$

or

$$-\frac{1+\tau_t^c}{1-\tau_t^w}\frac{\gamma c_t}{1-n_t} = -\frac{\gamma c_t}{1-n_t}$$

The optimal rates of tax are therefore $\tau_t^c = \tau_t^w = 0$, or $\tau_t^c = -\tau_t^w$.

Because the government must satisfy its budget constraint

$$g_t + (1+r)b_t = \tau_t^c c_t + \tau_t^w w_t n_t + b_{t+1}$$

the optimal solutions imply that it must either use pure debt finance or tax total household saving $w_t n_t - c_t$. The government budget constraint is then

$$g_t + (1+r)b_t = \tau_t(w_t n_t - c_t) + b_{t+1}$$

where $\tau_t = \tau_t^w = -\tau_t^c$. The implication is that, given these assumptions, optimal taxation requires that the excess of household labor income over consumption is taxed. This is not exactly the same as taxing savings as financial assets are not being taxed - at least explicitly.

Chapter 6

6.1. (a) Consider the following two-period OLG model. People consume in both periods but work only in period two. The inter-temporal utility of the representative individual in the first period is

$$\mathcal{U} = \ln c_1 + \beta[\ln c_2 + \alpha \ln(1 - n_2) + \gamma \ln g_2]$$

where c_1 and c_2 are consumption and k_1 (which is given) and k_2 are the stocks of capital in periods one and two, n_2 is work and g_2 is government expenditure in period two which is funded by a lump-sum tax in period two. Production in periods one and two are

$$\begin{aligned} y_1 &= Rk_1 = c_1 + k_2 \\ y_2 &= Rk_2 + \phi n_2 = c_2 + g_2 \end{aligned}$$

Find the optimal centrally-planned solution for c_1.

(b) Find the private sector solutions for c_1 and c_2, taking government expenditures as given.

(c) Compare the two solutions.

Solution

(a) The central planner maximizes $\mathcal{U}$ subject to the budget constraints for periods one and two with respect to the implicit budget constraint $g_2 = T_2$ with respect to c_1, c_2, n_2, k_2 and g_2. The Lagrangian is

$$\mathcal{L} = \ln c_1 + \beta[\ln c_2 + \alpha \ln(1 - n_2) + \gamma \ln g_2] + \lambda(Rk_1 - c_1 - k_2) + \mu(Rk_2 + \phi n_2 - c_2 - g_2)$$

The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_1} &= \frac{1}{c_1} - \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial c_2} &= \frac{\beta}{c_2} - \mu = 0 \\
\frac{\partial \mathcal{L}}{\partial n_2} &= -\frac{\alpha\beta}{1-n_2} + \mu\phi = 0 \\
\frac{\partial \mathcal{L}}{\partial k_2} &= -\lambda + \mu R = 0 \\
\frac{\partial \mathcal{L}}{\partial g_2} &= \frac{\beta\gamma}{g_2} - \mu = 0
\end{aligned}$$

Hence $c_2 = \beta R c_1 = \frac{\phi}{\alpha}(1-n_2) = \frac{1}{\gamma} g_2$. And from the budget constraint $k_2 = c_1 - Rk_1$. It remains therefore to solve c_1 and c_2 in terms of k_1. It can be shown that

$$\begin{aligned}
c_1 &= \frac{\phi + R^2 k_1}{[1+\beta(1+\alpha+\gamma)]R} \\
c_2 &= \frac{\beta(\phi + R^2 k_1)}{1+\beta(1+\alpha+\gamma)}
\end{aligned}$$

(b) The private sector maximizes

$$\mathcal{U} = \ln c_1 + \beta[\ln c_2 + \alpha \ln(1-n_2)]$$

subject to its budget constraint which can be derived from the national income identities and the government budget constraint $g_2 = T_2$. The Lagrangian for this problem is

$$\mathcal{L} = \ln c_1 + \beta[\ln c_2 + \alpha \ln(1-n_2)] + \lambda(Rk_1 - c_1 - k_2) + \mu(Rk_2 + \phi n_2 - c_2 - g_2)$$

where g_2 could be replaced with T_2. The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_1} &= \frac{1}{c_1} - \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial c_2} &= \frac{\beta}{c_2} - \mu = 0 \\
\frac{\partial \mathcal{L}}{\partial n_2} &= -\frac{\alpha\beta}{1-n_2} + \mu\phi = 0 \\
\frac{\partial \mathcal{L}}{\partial k_2} &= -\lambda + \mu R = 0
\end{aligned}$$

Hence, once again, $c_2 = \beta R c_1 = \frac{\phi}{\alpha}(1-n_2)$ and $k_2 = c_1 - Rk_1$.

(c) Thus the central planner chooses the same solution as the private sector. The difference is that the central planner also chooses $g_2 = \gamma c_2$. The reason for the similarities of the solutions is because lump-sum taxes are not distorting.

6.2 Suppose that in Exercise 6.1 the government finances its expenditures with taxes both on labor and capital in period two so that the government budget constraint is

$$g_2 = \tau_2 \phi n_2 + (R - R_2)k_2$$

where R_2 is the after-tax return to capital and τ_2 is the rate of tax of labor in period two. Derive the centrally-planned solutions for c_1 and c_2.

Solution

The Lagrangian for the central planner's problem is

$$\begin{aligned}\mathcal{L} &= \ln c_1 + \beta[\ln c_2 + \alpha \ln(1 - n_2) + \gamma \ln g_2] + \lambda(Rk_1 - c_1 - k_2) + \mu(Rk_2 + \phi n_2 - c_2 - g_2) \\ &\quad +\nu[g_2 - \tau_2 \phi n_2 - (R - R_2)k_2]\end{aligned}$$

The first-order conditions are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial c_1} &= \frac{1}{c_1} - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial c_2} &= \frac{\beta}{c_2} - \mu = 0 \\ \frac{\partial \mathcal{L}}{\partial n_2} &= -\frac{\alpha\beta}{1 - n_2} + \mu\phi - \nu\tau_2\phi = 0 \\ \frac{\partial \mathcal{L}}{\partial k_2} &= -\lambda + \mu R - \nu(R - R_2) = 0 \\ \frac{\partial \mathcal{L}}{\partial g_2} &= \frac{\beta\gamma}{g_2} - \mu = 0\end{aligned}$$

These equations, together with the constraints, form the required necessary conditions. The solutions for c_1 and c_2 can be shown to be the same as in Exercise 6.1.

6.3. (a) Continuing to assume that the government budget constraint is as defined in Exercise

6.2, find the private sector solutions for c_1 and c_2 when government expenditures and tax rates are pre-announced.

(b) Why may this solution not be time consistent?

Solution

(a) The private sector's solution is obtained by first eliminating g_2 to give the constraint

$$c_2 = R_2k_2 + (1-\tau_2)\phi n_2$$

The inter-temporal utility function is maximized subject to two constraints to give the Lagrangian

$$\begin{aligned}\mathcal{L} &= \ln c_1 + \beta[\ln c_2 + \alpha\ln(1-n_2)] + \lambda(Rk_1 - c_1 - k_2) \\ &\quad +\mu[R_2k_2 + (1-\tau_2)\phi n_2 - c_2]\end{aligned}$$

The first-order conditions are

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial c_1} &= \frac{1}{c_1} - \lambda = 0 \\ \frac{\partial\mathcal{L}}{\partial c_2} &= \frac{\beta}{c_2} - \mu = 0 \\ \frac{\partial\mathcal{L}}{\partial n_2} &= -\frac{\alpha\beta}{1-n_2} + \mu(1-\tau_2)\phi = 0 \\ \frac{\partial\mathcal{L}}{\partial k_2} &= -\lambda + \mu R_2 = 0 \\ \frac{\partial\mathcal{L}}{\partial g_2} &= \frac{\beta\gamma}{g_2} - \mu = 0\end{aligned}$$

The solutions are $c_2 = \beta Rc_1 = \frac{(1-\tau_2)\phi}{\alpha}(1-n_2)$ and $k_2 = c_1 - Rk_1$. Hence,

$$\begin{aligned}c_1 &= \frac{RR_2k_1 + (1-\tau_2)\phi}{R_2 + \beta R(1+\alpha)} \\ c_2 &= \beta R\frac{RR_2k_1 + (1-\tau_2)\phi}{R_2 + \beta R(1+\alpha)}\end{aligned}$$

These solutions are different from the solutions for c_1 and c_2 in Exercises 6.1 and 6.2.

(b) The solutions depend on the rates of labor tax τ_2 and capital tax $R - R_2$. As a result, if it is optimal for government to change these rates of tax in period two, this would affect the above solution which would not therefore be time consistent.

6.4 For Exercise 6.3 assume now that the government optimizes taxes in period two taking k_2 as given as it was determined in period one.

(a) Derive the necessary conditions for the optimal solution.

(b) Show that the optimal labor tax when period two arrives is zero. Is it optimal to taxe capital in period two?

Solution

(a) The government's problem is to maximize

$$\mathcal{U} = \ln c_2 + \alpha \ln(1 - n_2) + \gamma \ln g_2$$

subject to the two constraints

$$\begin{aligned} Rk_2 + \phi n_2 &= c_2 + g_2 \\ g_2 &= \tau_2 \phi n_2 + (R - R_2) k_2 \end{aligned}$$

Taking period one values and k_2 as given, the Lagrangian is

$$\begin{aligned} \mathcal{L} &= \ln c_2 + \alpha \ln(1 - n_2) + \gamma \ln g_2 + \lambda[Rk_2 + \phi n_2 - c_2 - g_2] \\ &\quad + \mu[g_2 - \tau_2 \phi n_2 - (R - R_2) k_2] \end{aligned}$$

The first-order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_2} &= \frac{1}{c_2} - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial n_2} &= -\frac{\alpha}{1 - n_2} + \lambda \phi - \mu \tau_2 \phi = 0 \\ \frac{\partial \mathcal{L}}{\partial g_2} &= \frac{\gamma}{g_2} - \mu = 0 \end{aligned}$$

The first-order conditions together with the constraints provide the necessary conditions for the solution which has no closed form as the equations are non-linear.

(b) Imposing a labor tax would be distorting. But taxing capital would not be distorting as k_2 is given and so cannot be affected by the capital tax. The capital tax, in effect, taxes the rents on capital. We also note that, comparing the first-order conditions for c_2 and n_2 in part (a) with those in Exercises 6.1 and 6.2, if $\tau_2 = 0$ and $\beta = R_2$ then we obtain the same relation between c_2 and n_2, namely, $c_2 = \frac{\phi}{\alpha}(1 - n_2)$. From the government budget constraint the optimal rate of capital taxation is $R - R_2 = \frac{g_2}{k_2}$.

6.5. Consider the following two-period OLG model in which each generation has the same number of people, N. The young generation receives an endowment of x_1 when young and $x_2 = (1 + \phi)x_1$ when old, where ϕ can be positive or negative. The endowments of the young generation grow over time at the rate γ. Each unit of saving (by the young) is invested and produces $1+\mu$ units of output ($\mu > 0$) when they are old. Each of the young generation maximizes $\ln c_{1t} + \frac{1}{1+r} \ln c_{2,t+1}$, where c_{1t} is consumption when young and $c_{2,t+1}$ is consumption when old.

(a) Derive the consumption and savings of the young generation and the consumption of the old generation.

(b) How do changes in ϕ, μ, r and γ affect these solutions?

(c) If $\phi = \mu$ how does this affect the solution?

Solution

(a) Let s_1 denote the savings of the young generation. The budget constraint when young in period t is

$$c_{1t} + s_1 = x_1$$

and when old in period $t + 1$ is

$$\begin{aligned} c_{2,t+1} &= x_2 + (1 + \mu)s_1 \\ &= (1 + \phi)x_1 + (1 + \mu)[x_1 - c_{1t}] \end{aligned}$$

This provides an inter-temporal budget constraint. The problem, therefore, is to maximize inter-temporal utility subject to the inter-temporal budget constraint.

The Lagrangian is

$$\mathcal{L} = \ln c_{1t} + \frac{1}{1+r}\ln c_{2,t+1} + \lambda[c_{2,t+1} - (1+\phi)x_1 - (1+\mu)[x_1 - c_{1t}]$$

The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_{1t}} &= \frac{1}{c_{1t}} + \lambda(1+\mu) = 0 \\
\frac{\partial \mathcal{L}}{\partial c_{2,t+1}} &= \frac{1}{1+r}\frac{1}{c_{2,t+1}} + \lambda = 0
\end{aligned}$$

and the Euler equation is

$$\frac{c_{2,t+1}}{c_{1t}}\frac{1+r}{1+\mu} = 1$$

It follows from the inter-temporal budget constraint and the Euler equation that consumption of the young in period t and when old in period $t+1$ are

$$\begin{aligned}
c_{1t} &= \frac{(2+\phi+\mu)(1+r)}{(1+\mu)(2+r)}x_1 \\
&= \frac{1+\frac{1+\phi}{1+\mu}}{1+\frac{1}{1+r}}x_1 \\
c_{2,t+1} &= \frac{1+\mu}{1+r}c_{1t} \\
&= \frac{2+\phi+\mu}{2+r}x_1
\end{aligned}$$

Because the endowment of the young increases over time at the rate γ, the general solution for period $t+i$ $(i \geq 0)$ is

$$\begin{aligned}
c_{1,t+i} &= \frac{(1+\gamma)^i(2+\phi+\mu)(1+r)}{(1+\mu)(2+r)}x_1 \\
c_{2,t+i+1} &= \frac{1+\mu}{1+r}c_{1,t+i} = \frac{(1+\gamma)^i(2+\phi+\mu)}{2+r}x_1
\end{aligned}$$

(b) It follows that the greater is ϕ, the larger is consumption in both periods as resources are greater. The greater is μ, the smaller is consumption when young but the larger is consumption when old. In contrast, the greater is r, the larger is consumption when young and the smaller is consumption when old. Thus changes in ϕ and r cause inter-temporal substitutions in different directions. γ determines the rate of growth of consumption over time both when young and old as it increases resources over time.

(c) If $\phi = \mu$ then consumption of the young in period t and when old in period $t+1$ are

$$\begin{aligned} c_{1t} &= \frac{2(1+r)}{(2+r)} x_1 \\ c_{2,t+1} &= \frac{2(1+\mu)}{2+r} x_1 \end{aligned}$$

Hence, consumption when young is now unaffected by either ϕ or μ, but consumption when old is now increased by a larger $\phi = \mu$.

Chapter 7

7.1. An open economy has the balance of payments identity

$$x_t - Qx_t^m + r^* f_t = \Delta f_{t+1}$$

where x_t is exports, x_t^m is imports, f_t is the net holding of foreign assets, Q is the terms of trade and r^* is the world rate of interest. Total output y_t is either consumed at home c_t^h or is exported, thus

$$y_t = c_t^h + x_t.$$

Total domestic consumption is c_t; y_t and x_t are exogenous.

(a) Derive the Euler equation that maximises $\sum_{s=0}^{\infty} \beta^s \ln c_{t+s}$ with respect to $\{c_t, c_{t+1},...; f_{t+1}, f_{t+2},...\}$ where $\beta = \frac{1}{1+\theta}$.

(b) Explain how and why the relative magnitudes of r^* and θ affect the steady-state solutions of c_t and f_t.

(c) Explain how this solution differs from that of the corresponding closed-economy.

(d) Comment on whether there are any benefits to being an open economy in this model.

(e) Obtain the solution for the current account.

(f) What are the effects on the current account and the net asset position of a permanent increase in x_t?

Solution

(a) The main difference between this problem and the basic problem of Chapter 7 is that output is exogenous. Consequently the national income identity is

$$y_t = c_t + x_t - Qx_t^m$$

and the balance of payments may be re-written as

$$y_t - c_t + r^* f_t = \Delta f_{t+1}.$$

The Lagrangian is therefore

$$\mathcal{L} = \sum_{s=0}^{\infty} \{\beta^s \ln c_{t+s} + \lambda_{t+s} [y_{t+s} - c_{t+s} - f_{t+s+1} + (1+r^*) f_{t+s}]\}.$$

The first-order conditions with respect to $\{c_{t+s},\, f_{t+s+1}; s \geq 0\}$ are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_{t+s}} &= \beta^s \frac{1}{c_{t+s}} - \lambda_{t+s} = 0 \qquad s \geq 0 \\ \frac{\partial \mathcal{L}}{\partial f_{t+s}} &= \lambda_{t+s}(1+r^*) - \lambda_{t+s-1} = 0 \qquad s > 0 \end{aligned}$$

plus the balance of payments. The open-economy Euler equation is therefore

$$\frac{\beta c_t}{c_{t+1}}(1+r^*) = 1.$$

(b) The Euler equation may be re-written as

$$\frac{c_{t+1}}{c_t} = \frac{1+r^*}{1+\theta}$$

which shows how optimal consumption will evolve in the future. Optimal consumption will therefore grow, stay constant or decline according as $r^* \gtreqless \theta$.

From the balance of payments the optimal net asset position therefore evolves as

$$\begin{aligned} f_{t+1} &= y_t - c_t + (1+r^*) f_t \\ &= \Sigma_{i=0}^{s} (1+r^*)^{s-i} y_{t+i} - (1+r^*) \Sigma_{i=0}^{s} \beta^i c_t + (1+r^*) f_t. \end{aligned}$$

Whether net assets grow, stay constant or decline depends on the future behavior of y_t and on whether $r^* \gtreqless \theta$. Consider the leading case where $y_{t+s} = y_t$ and $r^* = \theta$. There is then a steady-state level of net assets which is

$$f = f_t = \frac{y_{t+1} - c_{t+1}}{r^*}.$$

In other words,

$$c_t = y_t + r^* f_t.$$

If $r^* \gtrless \theta$ then a steady state solution requires that y_t grows at the same rate as consumption, namely, $r^* - \theta$.

(c) In a closed economy the net foreign asset position f_t is replaced in the steady-state solution for consumption by the stock of domestic assets.

(d) A benefit of being an open rather than a closed economy is that the stock of domestic assets must be non-negative whereas net foreign assets may be positive or negative. Having negative foreign assets implies that the economy is borrowing from abroad. This allows an economy to finance a negative current account balance, and hence better smooth consumption, than an economy that has to correct its current account deficit - possibly sharply - by reducing consumption through a cut in imports. This was a common feature of the Bretton Woods period when there were controls on foreign capital movements and policy was "stop-go" which increased the volatility of business cycle fluctuations.

(e) The current account is

$$\begin{aligned} ca_t &= x_t - Qx_t^m + r^* f_t \\ &= y_t - c_t + r^* f_t. \end{aligned}$$

This is also the steady-state solution for the current account described above.

(f) A permanent increase in exports x_t is equivalent in its effect on consumption to a permanent increase in income. This causes a permanent increase in consumption, but does not affect the current account as there is also a permanent increase in imports.

7.2. Consider two countries which consume home and foreign goods $c_{H,t}$ and $c_{F,t}$. Each period the home country maximizes

$$U_t = \left[c_{H,t}^{\frac{\sigma-1}{\sigma}} + c_{F,t}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

and has an endowment of y_t units of the home produced good. The foreign country is identical and its variables are denoted with an asterisk. Every unit of a good that is transported abroad has a real resource cost equal to τ so that, in effect, only a proportion $1-\tau$ arrives at its destination. $P_{H,t}$ is the home price of the home good and $P^*_{H,t}$ is the foreign price of the home good. The corresponding prices of the foreign good are $P_{F,t}$ and $P^*_{F,t}$. All prices are measured in terms of a common unit of world currency.

(a) If goods markets are competitive what is the relation between the four prices and how are the terms of trade in each country related?

(b) Derive the relative demands for home and foreign goods in each country.

(c) Hence comment on the implications of the presence of transport costs.

Note: This Exercise and the next, Exercise 7.3, is based on Obstfeld and Rogoff (2000).

Solution

(a) If markets are competitive and there are costs to trade which are borne by the importer (i.e. with producer pricing) then the home price of the home good is lower than its price abroad. If goods arbitrage holds this implies that

$$\begin{aligned} P_{H,t} &= (1-\tau)P^*_{H,t} \\ (1-\tau)P_{F,t} &= P^*_{F,t}. \end{aligned}$$

Given that the terms of trade for the home and foreign economies is $Q_{T,t} = \frac{P_{F,t}}{P_{H,t}}$ and $Q^*_{T,t} = \frac{P^*_{F,t}}{P^*_{H,t}}$, respectively, it follows that

$$Q^*_{T,t} = (1-\tau)^2 Q_{T,t}$$

(b) Each country maximizes utility in period t subject to their budget constraint. Ignoring all assets, as the problem is for one period, the budget constraint for the home country is

$$P_{H,t}y_t = P_{H,t}c_{H,t} + P_{F,t}c_{F,t}$$

which can be re-written as

$$y_t = c_{H,t} + Q_{T,t}c_{F,t}$$

where $Q_{T,t} = \frac{P_{F,t}}{P_{H,t}}$ is the terms of trade.

The Lagrangian for the home country is therefore

$$\mathcal{L} = \left[c_{H,t}^{\frac{\sigma-1}{\sigma}} + c_{F,t}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} + \lambda_t (y_t - c_{H,t} - Q_{T,t} c_{F,t})$$

and the first-order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_{H,t}} &= \left[\frac{U_t}{c_{H,t}} \right]^{\frac{1}{\sigma}} - \lambda_t = 0 \\ \frac{\partial \mathcal{L}}{\partial c_{F,t}} &= \left[\frac{U_t}{c_{F,t}} \right]^{\frac{1}{\sigma}} - \lambda_t Q_{T,t} = 0 \end{aligned}$$

Hence,

$$\frac{c_{H,t}}{c_{F,t}} = Q_{T,t}^{\sigma} = \left(\frac{P_{F,t}}{P_{H,t}} \right)^{\sigma}$$

For the foreign country the corresponding expression is

$$\begin{aligned} \frac{c_{H,t}^*}{c_{F,t}^*} &= Q_{T,t}^{*\sigma} = \left(\frac{P_{F,t}^*}{P_{H,t}^*} \right)^{\sigma} \\ &= (1-\tau)^{2\sigma} Q_{T,t}^{\sigma} \end{aligned}$$

(c) First, from $Q_{T,t}^* = (1-\tau)^2 Q_{T,t}$, the greater is τ, the more the terms of trade differ between the two countries. If there are no transport costs then the terms of trade are the same.

Second, as the ratio of expenditures on domestic goods to imports in each country is

$$\left(\frac{c_{H,t}}{Q_{T,t} c_{F,t}} \right) / \left(\frac{c_{F,t}^*}{Q_{T,t}^* c_{H,t}^*} \right) = (1-\tau)^{(1-\sigma)},$$

the greater is τ, the higher is the proportion of total expenditures on domestic goods. Obstfeld and Rogoff suggest that this may help explain home bias in consumption.

7.3. Suppose the model in Exercise 7.2 is modified so that there are two periods and intertemporal utility is

$$V_t = U(c_t) + \beta U(c_{t+1})$$

where $c_t = \left[c_{H,t}^{\frac{\sigma-1}{\sigma}} + c_{F,t}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$. Endowments in the two periods are y_t and y_{t+1}. Foreign prices $P^*_{H,t}$ and $P^*_{F,t}$ and the world interest rate are assumed given. The first and second period budget constraints are

$$\begin{aligned} P_{H,t}y_t + B &= P_{H,t}c_{H,t} + P_{F,t}c_{F,t} = P_tc_t \\ P_{H,t+1}y_{t+1} - (1+r^*)B &= P_{H,t+1}c_{H,t+1} + P_{F,t+1}c_{F,t+1} = P_{t+1}c_{t+1}, \end{aligned}$$

where P_t is the general price level, B is borrowing from abroad in world currency units in period t and r^* is the foreign real interest rate. It is assumed that there is zero foreign inflation.

(a) Derive the optimal solution for the home economy, including the domestic price level P_t.

(b) What is the domestic real interest rate r? Does real interest parity exist?

(c) How is r related to τ?

Solution

(a) The budget constraints imply that the indebted country repays its debt in period $t+1$ and hence must export home production in period $t+1$. Eliminating B from the budget constraints gives the two-period inter-temporal constraint for the home country

$$P_{H,t+1}y_{t+1} + (1+r^*)P_{H,t}y_t = P_{H,t+1}c_{H,t+1} + P_{F,t+1}c_{F,t+1} + (1+r^*)(P_{H,t}c_{H,t} + P_{F,t}c_{F,t}).$$

We now maximise V_t subject to this constraint.

The Lagrangian is

$$\begin{aligned} \mathcal{L} &= U\left\{\left[c_{H,t}^{\frac{\sigma-1}{\sigma}} + c_{F,t}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}\right\} + \beta U\left\{\left[c_{H,t+1}^{\frac{\sigma-1}{\sigma}} + c_{F,t+1}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}\right\} \\ &\quad +\lambda[P_{H,t+1}y_{t+1} + (1+r^*)P_{H,t}y_t - P_{H,t+1}c_{H,t+1} - P_{F,t+1}c_{F,t+1} \\ &\quad -(1+r^*)(P_{H,t}c_{H,t} + P_{F,t}c_{F,t})] \end{aligned}$$

and the first-order conditions for $i = H, F$ are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_{i,t}} &= \left[\frac{U_t}{c_{i,t}}\right]^{\frac{1}{\sigma}} - \lambda(1+r^*)P_{i,t} = 0 \\ \frac{\partial \mathcal{L}}{\partial c_{i,t+1}} &= \beta\left[\frac{U_{t+1}}{c_{i,t+1}}\right]^{\frac{1}{\sigma}} - \lambda P_{i,t+1} = 0. \end{aligned}$$

Hence the relative expenditures on home and foreign goods is

$$\frac{P_{H,t}c_{H,t}}{P_{F,t}c_{F,t}} = \left(\frac{P_{H,t}}{P_{F,t}}\right)^{1-\sigma}.$$

From total expenditure

$$\begin{aligned} 1 &= \frac{P_{H,t}c_{H,t}}{P_t c_t} + \frac{P_{F,t}c_{F,t}}{P_t c_t} \\ &= \frac{P_{F,t}c_{F,t}}{P_t c_t}\left[1 + \frac{P_{H,t}c_{H,t}}{P_{F,t}c_{F,t}}\right] \\ &= \frac{P_{F,t}c_{F,t}}{P_t c_t}\left[1 + \left(\frac{P_{H,t}}{P_{F,t}}\right)^{1-\sigma}\right]. \end{aligned}$$

Hence,

$$\frac{c_{F,t}}{c_t} = \frac{P_{F,t}^{1-\sigma}}{P_{H,t}^{1-\sigma} + P_{F,t}^{1-\sigma}}\frac{P_t}{P_{F,t}}.$$

From the consumption index

$$\begin{aligned} 1 &= \left[\frac{c_{H,t}}{c_t}\right]^{\frac{\sigma-1}{\sigma}} + \left[\frac{c_{F,t}}{c_t}\right]^{\frac{\sigma-1}{\sigma}} \\ &= \left[\frac{P_{H,t}^{1-\sigma}}{P_{H,t}^{1-\sigma} + P_{F,t}^{1-\sigma}}\frac{P_t}{P_{H,t}}\right]^{\frac{\sigma-1}{\sigma}} + \left[\frac{P_{F,t}^{1-\sigma}}{P_{H,t}^{1-\sigma} + P_{F,t}^{1-\sigma}}\frac{P_t}{P_{F,t}}\right]^{\frac{\sigma-1}{\sigma}}. \end{aligned}$$

Thus

$$P_t = \left[P_{H,t}^{1-\sigma} + P_{F,t}^{1-\sigma}\right]^{\frac{1}{1-\sigma}}.$$

The solution for total consumption is obtain from the Euler equation for c_t

$$\frac{\beta U'(c_{t+1})(1+r^*)}{U'(c_t)} = 1$$

and the inter-temporal budget constraint

$$P_{H,t+1}y_{t+1} + (1+r^*)P_{H,t}y_t = P_{t+1}c_{t+1} + (1+r^*)P_t c_t.$$

(b) If r^* is the foreign real interest rate and r is the domestic real interest rate then nominal interest parity implies that is

$$1 + r = (1+r^*)\frac{P_t}{P_{t+1}}$$

i.e. there is real interest parity only if domestic inflation is also zero.

(c) If the home country borrows in period t and repays in period $t+1$ then it must export part of its production in period $t+1$. As a result, $P_{H,t} = (1-\tau)P^*_{H,t}$ but $P^*_{H,t+1} = (1-\tau)P_{H,t+1}$. As foreign prices are constant

$$\begin{aligned} 1+r &= (1+r^*)\frac{P_t}{P_{t+1}} \\ &= (1+r^*)\frac{\left[P_{H,t}^{1-\sigma} + P_{F,t}^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}{\left[P_{H,t+1}^{1-\sigma} + P_{F,t+1}^{1-\sigma}\right]^{\frac{1}{1-\sigma}}} \\ &= (1+r^*)\frac{\left[\left((1-\tau)P^*_{H,t}\right)^{1-\sigma} + P_{F,t}^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}{\left[\left(\frac{P^*_{H,t}}{1-\tau}\right)^{1-\sigma} + P_{F,t}^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}. \end{aligned}$$

Hence, $\frac{\partial r}{\partial \tau} < 0$. Thus, the higher are transport costs, the lower is the domestic real interest rate.

7.4. Suppose the "world" is compromised of two similar countries where one is a net debtor. Each country consumes home and foreign goods and maximizes

$$V_t = \sum_{s=0}^{\infty} \beta^s \frac{(c^{\alpha}_{H,t+s}c^{1-\alpha}_{F,t+s})^{1-\sigma}}{1-\sigma}$$

subject to its budget constraint. Expressed in terms of home's prices, the home country budget constraint is

$$P_{H,t}c_{H,t} + S_t P_{F,t} c_{F,t} + \Delta B_{t+1} = P_{H,t} y_{H,t} + R_t B_t$$

where $c_{H,t}$ is consumption of home produced goods, $c_{F,t}$ is consumption of foreign produced goods, $P_{H,t}$ is the price of the home country's output which is denoted $y_{H,t}$ and is exogenous, $P_{F,t}$ is the price of the foreign country's output in terms of foreign prices, and B_t is the home country's borrowing from abroad expressed in domestic currency which is at the nominal rate of interest R_t and S_t is the nominal exchange rate. Interest parity is assumed to hold.

(a) Using an asterisk to denote the foreign country equivalent variable (e.g. $c^*_{H,t}$ is the foreign country's consumption of domestic output), what are the national income and balance of payments identities for the home country?

(b) Derive the optimal relative expenditure on home and foreign goods taking the foreign country - its output, exports and prices - and the exchange rate as given.

(c) Derive the price level P_t for the domestic economy assuming that $c_t = c_{H,t+s}^{\alpha} c_{F,t+s}^{1-\alpha}$.

(d) Obtain the consumption Euler equation for the home country.

(e) Hence derive the implications for the current account and the net foreign asset position. Comment on the implications of the home country being a debtor nation.

(f) Suppose that $y_t < y_t^*$ and both are constant, that there is zero inflation in each country, $R_t = R$ and $\beta = \frac{1}{1+R}$. Show that $c_t < c_t^*$ if $B_t \geq 0$.

Solution

(a) Home country's nominal national income identity is

$$P_{H,t} y_t = P_{H,t}(c_{H,t} + c_{F,t}^*).$$

Its nominal balance of payments is therefore

$$P_{H,t} c_{F,t}^* - S_t P_{F,t} c_{F,t} - R_t B_t = -\Delta B_{t+1}.$$

Assuming interest parity, the foreign country's balance of payments in foreign prices is therefore

$$P_{F,t} c_{F,t} - S_t^{-1} P_{H,t}^* c_{F,t} - R_t S_t^{-1} B_t = -S_t^{-1} \Delta B_{t+1}.$$

(b) The home country maximizes V_t subject to its budget constraint which can be written as

$$P_{H,t} y_t - P_{H,t} c_{H,t} - S_t P_{F,t} c_{F,t} - (1 + R_t) B_t + B_{t+1}.$$

The Lagrangian is therefore

$$\mathcal{L} = \sum_{s=0}^{\infty} \left\{ \beta^s \frac{(c_{H,t+s}^{\alpha} c_{F,t+s}^{1-\alpha})^{1-\sigma}}{1-\sigma} + \lambda_{t+s} [P_{H,t+s} y_{t+s} - P_{H,t+s} c_{H,t+s} - S_{t+s} P_{F,t+s} c_{F,t+s} - (1 + R_{t+s}) B_{t+s} + B_{t+s+1}] \right\}$$

and the first-order conditions for the home country are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_{H,t+s}} &= \beta^s \alpha c_{H,t+s}^{\alpha(1-\sigma)-1} c_{F,t+s}{}^{(1-\alpha)(1-\sigma)} - \lambda_{t+s} P_{H,t+s} = 0 \qquad s \geq 0 \\
\frac{\partial \mathcal{L}}{\partial c_{F,t+s}} &= \beta^s (1-\alpha) c_{H,t+s}^{\alpha(1-\sigma)} c_{F,t+s}{}^{(1-\alpha)(1-\sigma)-1} - \lambda_{t+s} S_{t+s} P_{F,t+s} = 0 \qquad s \geq 0 \\
\frac{\partial \mathcal{L}}{\partial B_{t+s}} &= -\lambda_{t+s}(1+R_{t+s}) + \lambda_{t+s-1} = 0 \qquad s > 0.
\end{aligned}$$

It follows that

$$\frac{c_{H,t}}{c_{F,t}} = \frac{\alpha}{1-\alpha} \frac{S_t P_{F,t}}{P_{H,t}}$$

where $\frac{S_t P_{F,t}}{P_{H,t}}$ is the terms of trade. The relative expenditure on home and foreign goods is therefore

$$\frac{P_{H,t} c_{H,t}}{S_t P_{F,t} c_{F,t}} = \frac{\alpha}{1-\alpha}.$$

(c) Total nominal expenditure on goods for the home economy is

$$P_t c_t = P_{H,t} c_{H,t} + S_t P_{F,t} c_{F,t}.$$

Hence

$$\frac{P_t c_t}{S_t P_{F,t} c_{F,t}} = \frac{1}{1-\alpha}$$

and the general price level is

$$P_t = \frac{P_{H,t}^{\alpha} \left(S_t P_{F,t}\right)^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}.$$

(d) Noting that $c_t = c_{H,t+s}^{\alpha} c_{F,t+s}^{1-\alpha}$ and that $P_t c_t = P_{H,t} c_{H,t} + S_t P_{F,t} c_{F,t}$ we can obtain

$$\frac{\partial \mathcal{L}}{\partial c_{t+s}} = \beta^s c_{t+s}^{-\sigma} - \lambda_{t+s} P_{t+s} = 0 \qquad s \geq 0$$

hence the Euler equation can be written as

$$\beta \left[\frac{c_t}{c_{t+1}}\right]^{\sigma} \frac{P_t}{P_{t+1}} (1+R_{t+1}) = 1.$$

(e) As the foreign country is the same

$$\frac{P_{H,t}^{*} c_{H,t}^{*}}{S_t^{-1} P_{F,t}^{*} c_{F,t}^{*}} = \frac{\alpha}{1-\alpha}$$

$P_{F,t} = P^*_{H,t}$ and $P_{H,t} = P^*_{F,t}$, the current account balance is

$$P_{H,t}c^*_{F,t} - S_t P_{F,t} c_{F,t} - R_t B_t = \frac{1-\alpha}{\alpha} S_t P^*_{H,t}(c^*_{H,t} - c_{H,t}) - R_t B_t = -\Delta B_{t+1}.$$

In addition, if the two countries are the same - including having the same levels of output - then $c^*_{H,t} = c_{H,t}$. Hence, in the absence of asymmetric country shocks, $B_t = 0$, i.e. net foreign assets are zero. But if, for example, the home country starts with net debt is repaying the debt, then

$$\begin{aligned} c_{H,t} &= c^*_{H,t} + \frac{\alpha}{1-\alpha}\frac{1}{S_t P^*_{H,t}}[\Delta B_{t+1} - R_t B_t] \\ &< c^*_{H,t}. \end{aligned}$$

If output is constant and the home country fails to maintain its consumption below that of the foreign country then it will have a permanent current deficit, its current account position will be unsustainable and its debt will accumulate. Home output growth could, however, make a permanent current account deficit sustainable.

(f) As there is zero inflation in each country, $R_t = R$ and $\beta = \frac{1}{1+R}$, it follows from the Euler equation that c_t is constant and hence $c_{H,t}$ and $c_{F,t}$ are also constant. The corresponding foreign variables are also constant. Moreover, all of the prices are constant.

From the home country's budget constraint

$$P_H y - Pc - (1+R)B_t = -B_{t+1}$$

or

$$\begin{aligned} B_t &= \frac{1}{1+R}[B_{t+1} + P_H y - Pc] \\ &= \Sigma_{i=0}^{\infty} \frac{P_H y - Pc}{(1+R)^{i+1}} \\ &= \frac{P_H y - Pc}{R}. \end{aligned}$$

Hence,

$$c = \frac{P_H}{P} y - R\frac{B_t}{P}$$

where

$$\frac{P_H}{P} = \alpha^{\alpha}(1-\alpha)^{1-\alpha}\left(\frac{P_H}{SP_H^*}\right)^{1-\alpha}$$

and

$$\begin{aligned} c^* &= \frac{P_H^*}{P^*}y^* + R\frac{S^{-1}B_t}{P^*} \\ \frac{P_H^*}{P^*} &= \alpha^{\alpha}(1-\alpha)^{1-\alpha}\left(\frac{P_H}{SP_H^*}\right)^{1-\alpha}. \end{aligned}$$

Thus $c^* > c$ if $B_t \geq 0$.

7.5. For the model described in Exercise 7.4, suppose that there is world central planner who maximizes the sum of individual country welfares:

$$W_t = \sum_{s=0}^{\infty}\beta^s\left[\frac{(c_{H,t+s}^{\alpha}c_{F,t+s}^{1-\alpha})^{1-\sigma}}{1-\sigma} + \frac{[(c_{H,t+s}^*)^{\alpha}(c_{F,t+s}^*)^{1-\alpha}]^{1-\sigma}}{1-\sigma}\right].$$

(a) What are the constraints in this problem?

(b) Derive the optimal world solution subject to these constraints where outputs and the exchange rate are exogenous.

(c) Comment on any differences with the solutions in Exercise 7.4.

Solution

(a) There are two national income identities

$$\begin{aligned} P_{H,t}y_t &= P_{H,t}(c_{H,t} + c_{F,t}^*) \\ P_{H,t}^*y_t^* &= P_{H,t}^*(c_{H,t}^* + c_{F,t}) \end{aligned}$$

the balance of payments

$$P_{H,t}c_{F,t}^* - S_tP_{F,t}c_{F,t} - R_tB_t = -\Delta B_{t+1}$$

and $P_{F,t} = P_{H,t}^*$ and $P_{H,t} = P_{F,t}^*$. Combining these through the balance of payments gives the single constraint

$$P_{H,t}y_t - P_{H,t}c_{H,t} - S_tP_{H,t}^*y_t^* + S_tP_{H,t}^*c_{H,t}^* - R_tB_t = -\Delta B_{t+1}.$$

(b) The Lagrangian is

$$\begin{aligned}
\mathcal{L} &= \sum_{s=0}^{\infty}\{\beta^s \left[\frac{[c_{H,t+s}^{\alpha}(y_{t+s}^* - c_{H,t+s}^*)^{1-\alpha}]^{1-\sigma}}{1-\sigma} + \frac{[(c_{H,t+s}^*)^{\alpha}(y_{t+s} - c_{H,t+s})^{1-\alpha}]^{1-\sigma}}{1-\sigma}\right] \\
&\quad +\lambda_{t+s}[P_{H,t+s}y_{t+s} - P_{H,t+s}c_{H,t+s} - S_{t+s}P_{H,t+s}^* y_{t+s}^* + S_{t+s}P_{H,t+s}^* c_{H,t+s}^* - (1+R_t)B_{t+s} + B_{t+s+1}]\}
\end{aligned}$$

which is maximized with respect to $c_{H,t+s}$, $c_{H,t+s}^*$ and B_{t+s}. The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_{H,t+s}} &= \beta^s \alpha c_{H,t+s}^{\alpha(1-\sigma)-1}(y_{t+s}^* - c_{H,t+s}^*)^{(1-\alpha)(1-\sigma)} - \beta^s(1-\alpha)(c_{H,t+s}^*)^{\alpha(1-\sigma)}(y_{t+s} - c_{H,t+s})^{(1-\alpha)(1-\sigma)-} \\
-\lambda_{t+s}P_{H,t+s} &= 0 \quad s \geq 0 \\
\frac{\partial \mathcal{L}}{\partial c_{H,t+s}^*} &= -\beta^s(1-\alpha)c_{H,t+s}^{\alpha(1-\sigma)}(y_{t+s}^* - c_{H,t+s}^*)^{(1-\alpha)(1-\sigma)-1} + \beta^s\alpha(c_{H,t+s}^*)^{\alpha(1-\sigma)-1}(y_{t+s} - c_{H,t+s})^{(1-\alpha)(1-} \\
+\lambda_{t+s}S_{t+s}P_{H,t+s}^* &= 0 \quad s \geq 0 \\
\frac{\partial \mathcal{L}}{\partial B_{t+s}} &= -\lambda_{t+s}(1+R_{t+s}) + \lambda_{t+s-1} = 0 \qquad s > 0.
\end{aligned}$$

It follows that

$$\frac{c_{H,t}}{c_{F,t}}\frac{-\frac{c_{H,t}^*}{c_{H,t}} + \frac{\alpha}{1-\alpha}\left(\frac{c_{H,t}^*}{c_{H,t}}\right)^{\alpha(1-\sigma)}\left(\frac{c_{F,t}}{c_{F,t}^*}\right)^{(1-\alpha)(1-\sigma)}}{\frac{c_{H,t}^*}{c_{H,t}} - \frac{1-\alpha}{\alpha}\left(\frac{c_{H,t}^*}{c_{H,t}}\right)^{\alpha(1-\sigma)}\left(\frac{c_{F,t}}{c_{F,t}^*}\right)^{(1-\alpha)(1-\sigma)}} = \frac{\alpha}{1-\alpha}\frac{S_t P_{H,t}^*}{P_{H,t}}$$

(c) Previously in Exercise 7.4 we obtained

$$\frac{c_{H,t}}{c_{F,t}} = \frac{\alpha}{1-\alpha}\frac{S_t P_{F,t}}{P_{H,t}} = \frac{\alpha}{1-\alpha}\frac{S_t P_{H,t}^*}{P_{H,t}}.$$

If the countries are identical then $c_{H,t} = c_{H,t}^*$ and $c_{F,t} = c_{F,t}^*$ and the two expressions are the same.

Chapter 8

8.1. Consider an economy in which money is the only financial asset, and suppose that households hold money solely in order to smooth consumption expenditures. The nominal household budget constraint for this economy is

$$P_t c_t + \Delta M_{t+1} = P_t y_t$$

where c_t is consumption, y_t is exogenous income, P_t is the price level and M_t is nominal money balances.

(a) If households maximize $\Sigma_{s=0}^{\infty} \beta^s U(c_{t+s})$, derive the optimal solution for consumption.

(b) Compare this solution with the special case where $\beta = 1$ and inflation is zero.

(c) Suppose that in (b) y_t is expected to remain constant except in period $t+1$ when it is expected to increase temporarily. Examine the effect on money holdings and consumption.

(d) Hence comment on the role of real balances in determining consumption in these circumstances.

Solution

(a) The Lagrangian for this problem can be written

$$\mathcal{L} = \sum_{s=0}^{\infty} \{\beta^s U(c_{t+s}) + \lambda_{t+s}[P_{t+s} y_{t+s} + M_{t+s} - M_{t+s+1} - P_{t+s} c_{t+s}]\}$$

where, for illustrative purposes, we have not used the *real* budget constraint. The first-order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_{t+s}} &= \beta^s U'_{t+s} - \lambda_{t+s} P_{t+s} = 0 \qquad s \geq 0 \\ \frac{\partial \mathcal{L}}{\partial M_{t+s}} &= \lambda_{t+s} - \lambda_{t+s-1} = 0 \qquad s > 0. \end{aligned}$$

This gives the consumption Euler equation

$$\frac{\beta U'_{t+1}}{U'_t} \frac{P_t}{P_{t+1}} = 1.$$

Assuming that $\beta = \frac{1}{1+\theta}$, and using $\pi_{t+1} = \frac{\Delta P_{t+1}}{P_t}$ and the approximation

$$\begin{aligned} \frac{U'{}_{t+1}}{U'{}_t} &\simeq 1 - \sigma_t \frac{\Delta c_{t+1}}{c_t} \\ \sigma_t &= -\frac{cU''_t}{U'_t} > 0 \end{aligned}$$

we obtain

$$\frac{\Delta c_{t+1}}{c_t} \simeq -\frac{\theta + \pi_{t+1}}{\sigma}.$$

Hence a steady-state in which consumption is constant requires a negative inflation rate of $\pi_{t+1} = -\theta$. As a result there would be a positive return to holding money, the only vehicle for savings in this economy.

(b) If $\beta = 1$ and $\pi_{t+1} = 0$ then $\pi_{t+1} = -\theta = 0$, which satisfies constant steady-state consumption. From the real budget constraint

$$c_t + (1 + \pi_{t+1})m_{t+1} - m_t = y_t$$

where $m_t = \frac{M_t}{P_t}$. Consumption therefore satisfies

$$c_t = y_t - m_{t+1} + m_t.$$

Hence a steady-state requires that $\Delta m_{t+1} = 0$ and $c_t = y_t$.

(c) If $y_{t+1} = y + \Delta y$ and $y_{t+s} = y$ for $s \neq 1$, then c_t, m_t and m_{t+1} are unaffected but

$$c_{t+1} = y + \Delta y - m_{t+2} + m_t.$$

Since optimal consumption is constant, an unexpected increase in income in period $t+1$ affects only the stock of money which increases permanently in period $t+2$ to $m_{t+2} = m_t + \Delta y$.

(d) This implies that if there is a zero discount rate and zero inflation, an increase in real balances has no effect on steady-state consumption.

8.2. Suppose that the nominal household budget constraint is

$$\Delta B_{t+1} + \Delta M_{t+1} + P_t c_t = P_t x_t + R_t B_t$$

where c_t is consumption, x_t is exogenous income, B_t is nominal bond holding, M_t is nominal money balances, P_t is the general price level, $m_t = M_t/P_t$ and R_t is a nominal rate of return.

(a) Derive the real budget constraint.

(b) Comment on whether or not this implies that money is super-neutral in the whole economy.

(c) If households maximize

$$V_t = \Sigma_{s=0}^{\infty} \beta^s U(c_{t+s}, m_{t+s})$$

where the utility function is

$$U(c_t, m_t) = \frac{\left[\frac{c_t^{\alpha} m_t^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\right]^{1-\sigma}}{1-\sigma}$$

obtain the demand for money.

Solution

(a) The real budget constraint is obtained by deflating the nominal budget constraint by the general price level P_t. The real budget constraint is therefore

$$\frac{P_{t+1}}{P_t}\frac{B_{t+1}}{P_{t+1}} + \frac{P_{t+1}}{P_t}\frac{M_{t+1}}{P_{t+1}} - \frac{M_t}{P_t} + c_t = x_t + (1+R_t)\frac{B_t}{P_t}$$

or

$$(1+\pi_{t+1})b_{t+1} + (1+\pi_{t+1})m_{t+1} - m_t + c_t = x_t + (1+R_t)b_t$$

where $b_t = \frac{B_t}{P_t}$, $m_t = \frac{M_t}{P_t}$ and $\pi_{t+1} = \frac{\Delta P_{t+1}}{P_t}$ is the inflation rate.

(b) Money is super-neutral if an increase in the nominal money supply has no real effects in steady state. The real budget constraint in steady state is

$$c = x + (R - \pi)b - \pi m.$$

This depends on a nominal variable - inflation - and appears to show that real consumption is reduced by the presence of an "inflation tax", πm. This is not, however, the full story as the inflation tax also enters the government budget constraint - as tax revenues - and reduces other taxes - such as lump-sum taxes - one-for-one. As a result, in full general equilibrium the inflation tax does not affect real consumption and so money is super-neutral in the economy as a whole. Money would not be super-neutral if, for example, there were additional costs associated with supplying money or with collecting the inflation tax. For further details see Chapter 8.

(c) We have shown in Chapter 8 that maximising $V_t = \sum_0^\infty \beta^s U(c_{t+s}, m_{t+s})$ subject to the real household budget constraint gives the Lagrangian

$$\mathcal{L} = \sum_{s=0}^{\infty} \left\{ \begin{array}{c} \beta^s U(c_{t+s}, m_{t+s}) + \lambda_{t+s}[x_{t+s} + (1 + R_{t+s})b_{t+s} + m_{t+s} \\ -(1 + \pi_{t+s+1})(b_{t+s+1} + m_{t+s+1}) - c_{t+s}] \end{array} \right\}$$

for which the first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_{t+s}} &= \beta^s U_{c,\,t+s} - \lambda_{t+s} = 0 \qquad s \geq 0 \\
\frac{\partial \mathcal{L}}{\partial b_{t+s}} &= \lambda_{t+s}(1 + R_{t+s}) - \lambda_{t+s-1}(1 + \pi_{t+s}) = 0 \qquad s > 0 \\
\frac{\partial \mathcal{L}}{\partial m_{t+s}} &= \beta^s U_{m,\,t+s} + \lambda_{t+s} - \lambda_{t+s-1}(1 + \pi_{t+s}) = 0 \qquad s > 0
\end{aligned}$$

which gives $U_{m,t+1} = U_{c,t+1} R_{t+1}$ from which we may obtain the long-run demand for money. For the utility function in this question this specializes to

$$\frac{(1-\alpha) c_{t+1}^{\alpha(1-\sigma)} m_{t+1}{}^{(1-\alpha)(1-\sigma)-1}}{[\alpha^\alpha (1-\alpha)^{1-\alpha}]^{1-\sigma}} = \frac{\alpha c_{t+1}^{\alpha(1-\sigma)-1} m_{t+1}{}^{(1-\alpha)(1-\sigma)} R_{t+1}}{[\alpha^\alpha (1-\alpha)^{1-\alpha}]^{1-\sigma}}.$$

Hence, the demand for money is

$$m_{t+1} = \frac{1-\alpha}{\alpha} \frac{c_{t+1}}{R_{t+1}}.$$

If bonds are risk free then R_{t+1} is known at time t.

8.3. Consider a cash-in-advance economy with the national income identity

$$y_t = c_t + g_t$$

and the government budget constraint

$$\Delta B_{t+1} + \Delta M_{t+1} + P_t T_t = P_t g_t + R_t B_t$$

where c_t is consumption, y_t is exogenous national income, B_t is nominal bond holding, M_t is nominal money balances, P_t is the general price level, $m_t = M_t/P_t$, T_t are lump-sum taxes, R_t is a nominal rate of return and the government consumes a random real amount $g_t = g + e_t$ where e_t is an independently and identically distributed random shock with zero mean.

(a) If households maximize $\Sigma_{s=0}^{\infty}\beta^s \ln c_{t+s}$ where $\beta = \frac{1}{1+\theta}$, derive the optimal solutions for consumption and money holding.

(b) Comment on how a positive government expenditure shock affects consumption and money holding.

(c) Is money super-neutral in this economy?

Solution

(a) Eliminating g_t and writing the cash-in advance constraint as $m_t = c_t$, enables the real household budget constraint to be written as

$$(1 + \pi_{t+1})b_{t+1} + (1 + \pi_{t+1})c_{t+1} + T_t = y_t + (1 + R_t)b_t.$$

The Lagrangian is therefore

$$\mathcal{L} = \sum_{s=0}^{\infty} \left\{ \begin{array}{c} \beta^s \ln c_{t+s} + \lambda_{t+s}[y_{t+s} + (1 + R_{t+s})b_{t+s} + \\ -(1 + \pi_{t+s+1})(b_{t+s+1} + c_{t+s+1}) - T_{t+s}] \end{array} \right\}.$$

The first-order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_{t+s}} &= \beta^s U_{c,\,t+s} - \lambda_{t+s-1}(1 + \pi_{t+s}) = 0 \qquad s \geq 0 \\ \frac{\partial \mathcal{L}}{\partial b_{t+s}} &= \lambda_{t+s}(1 + R_{t+s}) - \lambda_{t+s-1}(1 + \pi_{t+s}) = 0 \qquad s > 0. \end{aligned}$$

This gives the Euler equation

$$\frac{\beta c_t}{c_{t+1}} \frac{1 + R_t}{1 + \pi_{t+1}} = 1.$$

Hence consumption is constant in steady-state if $r_t = R_t - \pi_{t+1} = \theta$.

From the budget constraint, and assuming for convenience that R_t and π_t are constant with $r = R - \pi > 0$,

$$\begin{aligned} b_t &= \frac{1+\pi}{1+R}(b_{t+1} + c_{t+1}) + \frac{1}{1+R}(T_t - y_t) \\ &= \frac{c_t}{r} + \frac{1}{1+R}\Sigma_{s=0}^{\infty}\frac{T_{t+s} - y_{t+s}}{(1+r)^s}. \end{aligned}$$

Hence

$$c_t = m_t = \frac{r}{1+R}\Sigma_{s=0}^{\infty}\frac{y_{t+s} - T_{t+s}}{(1+r)^s} + rb_t$$

which is the permanent income from after-tax income plus interest earnings. In steady-state, therefore,

$$c = m = \frac{1}{1+\pi}(y - T) + rb.$$

(b) From the national income identity, a government expenditure shock e_t implies that

$$\begin{aligned} y_t &= c_t + g + e_t \\ &= y + (c_t - c) + e_t. \end{aligned}$$

If national income is fixed then consumption (and money holdings) fall as $c_t - c = -e_t$. But if national income increases then

$$c_t = m_t = c + \frac{r}{1+R}[c_t - c + e_t - (T_t - T)].$$

It follows that

$$c_t - c = \frac{r}{1+\pi}[e_t - (T_t - T)].$$

Thus if the temporary increase in government expenditures is tax financed then there is no effect on consumption or money holdings. But if it is not tax financed then consumption and money holdings increase by $c_t - c = \frac{r}{1+\pi}e_t$.

(c) An increase in the nominal money stock raises the general price level one-for-one in this economy leaving real money balances, consumption and output unaffected, and so money is super-neutral.

8.4. Suppose that some goods $c_{1,t}$ must be paid for only with money M_t and the rest $c_{2,t}$ are bought on credit L_t using a one period loan to be repaid at the start of next period at the nominal rate of interest $R+\rho$, where R is the rate of interest on bonds which are a savings vehicle. The prices of these goods are P_{1t} and P_{2t}. If households maximize $\Sigma_{s=0}^{\infty}(1+R)^{-s}U(c_{t+s})$ subject to their budget constraint, where $U(c_t)=\ln c_t$, $c_t=\frac{c_{1,t}^{\alpha}c_{2,t}^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}$, and income y_t is exogenous,

(a) derive the expenditures on cash purchases relative to credit.

(b) Obtain the optimal long-run solutions for $c_{1,t}$ and $c_{2,t}$ when exogenous income y_t is constant.

(c) Comment on the case where there is no credit premium.

Solution

(a) For cash-only goods there is a cash-in-advance constraint $M_t=P_{1t}c_{1t}$ and for credit goods $L_{t+1}=P_{2t}c_{2t}$. The nominal budget constraint is

$$\Delta M_{t+1}+\Delta B_{t+1}+(R+\rho)L_t+P_tc_t=P_ty_t+RB_t+\Delta L_{t+1}$$

and total expenditure is

$$P_tc_t=P_{1t}c_{1t}+P_{2t}c_{2t}.$$

The budget constraint can therefore be re-written as

$$B_{t+1}+P_{1,t+1}c_{1,t+1}+(1+R+\rho)P_{2,t-1}c_{2,t-1}=P_ty_t+(1+R)B_t.$$

The Lagrangian is

$$\begin{aligned}\mathcal{L} \quad &= \quad \sum\nolimits_{s=0}^{\infty}\{(1+R)^{-s}\ln\frac{c_{1,t}^{\alpha}c_{2,t}^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}+\lambda_{t+s}[P_{t+s}y_{t+s}+(1+R_{t+s})B_{t+s}\\ &\qquad -B_{t+s+1}-P_{1,t+s+1}c_{1,t+s+1}-(1+R+\rho)P_{2,t+s-1}c_{2,t+s-1}]\}.\end{aligned}$$

The first-order conditions are

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_{1,t+s}} &= (1+R)^{-s}\frac{\alpha}{c_{1,t+s}} - \lambda_{t+s-1}P_{1,t+s} = 0 \qquad s \geq 0 \\
\frac{\partial \mathcal{L}}{\partial c_{2,t+s}} &= (1+R)^{-s}\frac{1-\alpha}{c_{2,t+s}} - \lambda_{t+s+1}(1+R+\rho)P_{2,t+s} = 0 \qquad s \geq 0 \\
\frac{\partial \mathcal{L}}{\partial B_{t+s}} &= \lambda_{t+s}(1+R) - \lambda_{t+s-1} = 0 \qquad s > 0.
\end{aligned}
$$

The ratio of cash to credit is therefore

$$
\frac{M_t}{L_{t+1}} = \frac{P_{1t}c_{1t}}{P_{2t}c_{2t}} = \frac{\alpha}{1-\alpha}\frac{1+R+\rho}{(1+R)^2}.
$$

Thus, the larger the premium on credit ρ, or the lower the rate of interest R, the greater cash purchases relative to credit purchases.

(b) The Euler equations are

$$
\frac{P_{1,t+1}c_{1,t+1}}{P_{1t}c_{1t}} = \frac{P_{2,t+1}c_{2,t+1}}{P_{2t}c_{2t}} = 1.
$$

Hence nominal expenditures are constant.

From the budget constraint

$$
\begin{aligned}
B_t &= \frac{1}{1+R}[B_{t+1} + P_{1,t+1}c_{1,t+1} + (1+R+\rho)P_{2,t-1}c_{2,t-1} - P_t y_t] \\
&= \frac{1}{R}[P_{1t}c_{1t} + (1+R+\rho)P_{2t}c_{2t} - P_t y_t] \\
&= \frac{1}{R}[\frac{\alpha + (1-\alpha)(1+R)^2}{\alpha}P_{1t}c_{1t} - P_t y_t].
\end{aligned}
$$

Thus

$$
\begin{aligned}
P_{1t}c_{1t} &= \frac{\alpha}{\alpha + (1-\alpha)(1+R)^2}(P_t y_t + B_t) \\
P_{2t}c_{2t} &= \frac{(1-\alpha)(1+R)^2}{[\alpha + (1-\alpha)(1+R)^2](1+R+\rho)}(P_t y_t + B_t).
\end{aligned}
$$

(c) If there is no credit premium then

$$
\frac{M_t}{L_{t+1}/(1+R)} = \frac{\alpha}{1-\alpha}.
$$

The discounted cost of borrowing is therefore equal to the cost of using cash and so the shares of cash and credit just reflect the form of the consumption index.

8.5. Suppose that an economy can either use cash-in-advance or credit. Compare the long-run levels of consumption that result from these choices for the economy in Exercise 8.4 when there is a single consumption good c_t.

Solution

(i) Cash-in advance

The budget constraint is

$$B_{t+1} + P_{1,t+1}c_{t+1} = P_t y_t + (1+R)B_t$$

and the Lagrangian is

$$\begin{aligned}\mathcal{L} &= \sum\nolimits_{s=0}^{\infty} \{(1+R)^{-s} \ln c_{t+s} + \lambda_{t+s}[P_{t+s}y_{t+s} + (1+R_{t+s})B_{t+s} \\ &\quad -B_{t+s+1} - P_{t+s+1}c_{t+s+1}\}.\end{aligned}$$

The first-order conditions are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial c_{t+s}} &= (1+R)^{-s}\frac{1}{c_{t+s}} - \lambda_{t+s-1}P_{t+s} = 0 \qquad s \geq 0 \\ \frac{\partial \mathcal{L}}{\partial B_{t+s}} &= \lambda_{t+s}(1+R) - \lambda_{t+s-1} = 0 \qquad s > 0.\end{aligned}$$

The Euler equation is

$$\frac{P_{t+1}c_{t+1}}{P_t c_t} = 1.$$

Hence, from the budget constraint, the long-run level of consumption is

$$c_t = y + R\frac{B}{P}.$$

(ii) Credit

The budget constraint is now

$$B_{t+1} + (1 + R + \rho)P_{t-1}c_{t-1} = P_t y_t + (1 + R)B_t$$

and the Lagrangian is

$$\begin{aligned}\mathcal{L} &= \sum\nolimits_{s=0}^{\infty}\{(1+R)^{-s}\ln c_{t+s} + \lambda_{t+s}[P_{t+s}y_{t+s} + (1 + R_{t+s})B_{t+s} \\ &\quad -B_{t+s+1} - (1 + R + \rho)P_{t+s-1}c_{t+s-1}]\}.\end{aligned}$$

The first-order conditions are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial c_{t+s}} &= (1+R)^{-s}\frac{1-\alpha}{c_{t+s}} - \lambda_{t+s+1}(1 + R + \rho)P_{t+s} = 0 \qquad s \geq 0 \\ \frac{\partial \mathcal{L}}{\partial B_{t+s}} &= \lambda_{t+s}(1+R) - \lambda_{t+s-1} = 0 \qquad s > 0.\end{aligned}$$

The Euler equation is again

$$\frac{P_{t+1}c_{t+1}}{P_t c_t} = 1.$$

Hence from the budget constraint the long-run level of consumption is

$$c_t = \frac{y + R\frac{B}{P}}{1 + R + \rho} < y + R\frac{B}{P}.$$

It follows that consumption is lower when credit is used instead of cash and, the greater the credit premium, the larger the disparity.

8.6. Consider the following demand for money function which has been used to study hyper-inflation

$$m_t - p_t = -\alpha(E_t p_{t+1} - p_t), \;\; \alpha > 0$$

where M_t= nominal money, $m_t = \ln M_t$, P_t= price level and $p_t = \ln P_t$.

(a) Contrast this with a more conventional demand function for money, and comment on why it might be a suitable formulation for studying hyper-inflation?

(b) Derive the equilibrium values of p_t and the rate of inflation if the supply of money is given by

$$\Delta m_t = \mu + \varepsilon_t$$

where $\mu > 0$ and $E_t[\varepsilon_{t+1}] = 0$.

(c) What will be the equilibrium values of p_t if

(i) the stock of money is expected to deviate temporarily in period $t+1$ from this money supply rule and take the value m^*_{t+1},

(ii) the rate of growth of money is expected to deviate permanently from the rule and from period $t+1$ grow at the rate υ.

Solution

(a) A more conventional demand function for money includes the nominal interest rate as an argument rather than expected inflation. From the Fisher equation, the nominal interest rate equals the nominal interest rate plus expected inflation. In hyper-inflation expected inflation is very large relative to the real interest rate then the nominal interest rate can be closely approximated by expected inflation, as in the Cagan money demand function.

(b) The money demand function can be solved for the price level to give the forward-looking equation

$$p_t = \frac{\alpha}{1+\alpha} E_t p_{t+1} + \frac{1}{1+\alpha} m_t.$$

Solving forwards gives

$$p_t = \left(\frac{\alpha}{1+\alpha}\right)^n E_t p_{t+n} + \frac{1}{1+\alpha} \Sigma_{s=0}^{n-1} \left(\frac{\alpha}{1+\alpha}\right)^s E_t m_{t+s}.$$

Hence, as $n \to \infty$,

$$p_t = \frac{1}{1+\alpha} \Sigma_{s=0}^{\infty} \left(\frac{\alpha}{1+\alpha}\right)^s E_t m_{t+s}.$$

As $\Delta m_t = \mu + \varepsilon_t$

$$m_{t+s} = m_t + s\mu + \Sigma_{i=1}^{s} \varepsilon_{t+i}$$

and so

$$E_t m_{t+s} = m_t + s\mu$$

and, as $\Sigma_{s=0}^{\infty}\theta^s s = \frac{\theta}{(1-\theta)^2}$ for $|\theta| < 1$,

$$\begin{aligned} p_t &= \frac{1}{1+\alpha}\Sigma_{s=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^s (m_t + s\mu) \\ &= m_t + \mu. \end{aligned}$$

The rate of inflation is $\Delta p_t = \Delta m_t = \mu + \varepsilon_t$.

(c) (i) If $E_t m_{t+1} = m_{t+1}^*$, then

$$\begin{aligned} p_t &= \frac{1}{1+\alpha}\Sigma_{s=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^s E_t m_{t+s} \\ &= m_t + \alpha\mu + \frac{\alpha}{(1+\alpha)^2}[m_{t+1}^* - m_t - \mu] \end{aligned}$$

(ii) If $E_t m_{t+s} = m_t + \nu + \varepsilon_{t+1}$ for $\dot{s} > 0$ then

$$\begin{aligned} p_t &= \frac{1}{1+\alpha}\Sigma_{s=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^s E_t m_{t+s} \\ &= m_t + \alpha\nu. \end{aligned}$$

Chapter 9

9.1. Consider an economy that produces a single good in which households maximize

$$V_t = \sum_{s=0}^{\infty} \beta^s \left[\ln c_{t+s} - \phi \ln n_{t+s} + \gamma \ln \frac{M_{t+s}}{P_{t+s}} \right], \qquad \beta = \frac{1}{1+r}$$

subject to the nominal budget constraint

$$P_t c_t + \Delta B_{t+1} + \Delta M_{t+1} = P_t d_t + W_t n_t + R B_t.$$

where c consumption, n is employment, W is the nominal wage rate, d is total real firm net revenues distributed as dividends, B is nominal bond holdings, R is the nominal interest rate, M is nominal money balances, P is the price level and r is the real interest rate. Firms maximize the present value of nominal net revenues

$$\Pi_t = \sum_{s=0}^{\infty} (1+r)^{-s} P_{t+s} d_{t+s}$$

where $d_t = y_t - w_t n_t$, the real wage is $w_t = W_t / P_t$ and the production function is $y_t = A_t n_t^{\alpha}$.

(a) Derive the optimal solution on the assumption that prices are perfectly flexible.

(b) Assuming that inflation is zero, suppose that, following a shock, for example, to the money supply, firms are able to adjust their price with probability ρ, and otherwise price retains its previous value. Discuss the consequences for the expected price level following the shock.

(c) Suppose that prices are fully flexible but the nominal wage adjusts to shocks with probability ρ. What are the consequences for the economy?

Solution

The Lagrangian for households in real terms is

$$\mathcal{L} = \sum_{s=0}^{\infty} \left\{ \begin{array}{c} (1+r)^{-s} \left[\ln c_{t+s} - \phi \ln n_{t+s} + \gamma \ln m_{t+s} \right] \\ +\lambda_{t+s} [d_{t+s} + w_{t+s} n_{t+s} + (1+R) b_{t+s} + m_{t+s} - c_{t+s} - (1+\pi_{t+s+1}) b_{t+s+1} - (1+\pi_{t+s+1}) m_{t+s+1}] \end{array} \right\}$$

where $b_t = \frac{B_t}{P_t}$, $m_t = \frac{M_t}{P_t}$ and the inflation rate $\pi_{t+1} = \frac{\Delta P_{t+1}}{P_t}$. The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_{t+s}} &= (1+r)^{-s}\frac{1}{c_{t+s}} - \lambda_{t+s} = 0 \qquad s \geq 0 \\
\frac{\partial \mathcal{L}}{\partial n_{t+s}} &= -(1+r)^{-s}\frac{\phi}{n_{t+s}} + \lambda_{t+s}w_{t+s} = 0 \qquad s \geq 0 \\
\frac{\partial \mathcal{L}}{\partial m_{t+s}} &= (1+r)^{-s}\frac{\gamma}{m_{t+s}} - \lambda_{t+s} - \lambda_{t+s-1}(1+\pi_{t+s}) = 0 \qquad s > 0 \\
\frac{\partial \mathcal{L}}{\partial b_{t+s}} &= \lambda_{t+s}(1+R) - \lambda_{t+s-1}(1+\pi_{t+s}) = 0 \qquad s > 0.
\end{aligned}$$

The consumption Euler equation is

$$\frac{(1+r)^{-1}c_t}{c_{t+1}}\frac{1+R}{1+\pi_{t+1}} = \frac{c_t}{c_{t+1}} = 1$$

which implies that optimal consumption is constant if inflation is constant. The supply of labor is given by

$$n_t = \phi\frac{c_t}{w_t}$$

and will also be constant if w_t is constant. The demand for real money is

$$m_t = \gamma\frac{c_t}{1+R}.$$

The firm, maximizes nominal net revenues

$$\Pi_t = \sum_{s=0}^{\infty}(1+r)^{-s}P_{t+s}(A_{t+s}n_{t+s}^{\alpha} - w_{t+s}n_{t+s})$$

for which the first-order condition is

$$\frac{\partial \Pi_t}{\partial n_{t+s}} = (1+r)^{-s}P_{t+s}[\alpha\frac{y_{t+s}}{n_{t+s}} - w_{t+s}] = 0.$$

Hence the demand for labor is

$$n_t = \alpha\frac{y_t}{w_t}$$

and dividends are

$$d_t = y_t - w_t n_t = (1-\alpha)y_t.$$

In steady state the rate of inflation and all real variables are constant. From the real household budget constraint the solutions for consumption, labor and money are

$$
\begin{aligned}
c &= y + rb - \pi m \\
&= \frac{\phi}{\alpha}c + rb - \frac{\pi\gamma}{1+R}c \\
&= \frac{rb}{1 - \frac{\phi}{\alpha} + \frac{\pi\gamma}{1+R}}.
\end{aligned}
$$

It is then straightforward to find the steady-state values of the other variables.

The price level may be obtained from the money market equation $m = \gamma\frac{c}{1+R}$ when

$$P = \frac{1+R}{\gamma}\frac{M}{c}.$$

As markets clear and prices are perfectly flexible the long-run demands for goods and labor equal their long-run supplies in each period. This is the standard benchmark case.

(b) If prices are perfectly flexible, a shock to the level of the money supply will affect the price level and the nominal wage but not the real variables. But if prices are not perfectly flexible there will be real effects.

If firms can adjust their prices with a probability of ρ, which is less than unity then, if inflation is zero, the expected price level in period t following a temporary shock of ε to the money supply is

$$EP_t = \rho P_t^* + (1-\rho)P$$

where the fully flexible price P_t^* is

$$
\begin{aligned}
P_t^* &= \alpha\frac{y}{n}\frac{1+R}{\gamma}\frac{M_t}{c} \\
&= P + \frac{1+R}{\gamma}\frac{\varepsilon}{c}.
\end{aligned}
$$

Hence the expected price level under imperfect price flexibility is

$$EP_t = P + \frac{(1-\rho)(1+R)}{\gamma}\frac{\varepsilon}{c}$$

where $EP_t > P$ but $EP_t < P_t^*$.

(c) If prices are fully flexible but the nominal wage is not then the expected nominal wage is

$$EW_t = \rho W_t^* + (1-\rho)W$$

where the fully flexible nominal wage W_t^* is

$$\begin{aligned} W_t^* &= \phi \frac{P_t^* c}{n} = \frac{\phi(1+R)}{\gamma}\frac{M_t}{n} \\ &= W + \frac{\phi(1+R)}{\gamma}\frac{\varepsilon}{n}. \end{aligned}$$

Hence the expected nominal wage is

$$EW_t = W + \frac{(1-\rho)\phi(1+R)}{\gamma}\frac{\varepsilon}{n}$$

where $EW_t > W$ but $EW_t < W_t^*$. This suggests that the expected real wage $\frac{EW_t}{P_t^*}$ under imperfect wage flexibility will be lower than the fully flexible real wage. This would create an incentive for firms to increase output and employment but would inhibit the supply of labor. In this way monetary policy could have a temporary real effect on the economy.

9.2. Consider an economy where prices are determined in each period under imperfect competition in which households have the utility function

$$U[c_t, n_t(i)] = \ln c_t - \eta \ln n_t(i)$$

with $i = 1, 2$. Total household consumption c_t is obtained from the two consumption goods $c_t(1)$ and $c_t(2)$ through the aggregator

$$c_t = \frac{c_t(1)^{\phi} c_t(2)^{1-\phi}}{\phi^{\phi}(1-\phi)^{1-\phi}}$$

and $n_t(1)$ and $n_t(2)$ are the employment levels in the two firms which have production functions

$$y_t(i) = A_{it} n_t(i)$$

and profits

$$\Pi_t(i) = P_t(i)y_t(i) - W_t(i)n_t(i)$$

where $P_t(i)$ is the output price and $W_t(i)$ is the wage rate paid by firm i. If total consumption expenditure is

$$P_t c_t = P_t(1)c_t(1) + P_t(2)c_t(2)$$

(a) Derive the optimal solutions for the household, treating firm profits as exogenous.

(b) Show how the price level for each firm is related to the common wage W_t and comment on your result.

Solution

(a) The household budget constraint is

$$\begin{aligned} P_t c_t &= P_t(1)c_t(1) + P_t(2)c_t(2) \\ &= W_t(i)n_t(i) + \Pi_t(1) + \Pi_t(2) \end{aligned}$$

if each household holds an equal share in each firm. Households maximize utility in each period using the Lagrangian

$$\begin{aligned} \mathcal{L} &= \ln\left(\frac{c_t(1)^{\phi} c_t(2)^{1-\phi}}{\phi^{\phi}(1-\phi)^{1-\phi}}\right) - \eta \ln n_t(i) \\ &\quad + \lambda_t[W_t(i)n_t(i) + \Pi_t(1) + \Pi_t(2) - P_t(1)c_t(1) - P_t(2)c_t(2)]. \end{aligned}$$

The first-order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t(1)} &= \phi \frac{1}{c_t(1)} - \lambda_t P_t(1) = 0 \\ \frac{\partial \mathcal{L}}{\partial c_t(2)} &= (1-\phi)\frac{1}{c_t(2)} - \lambda_t P_t(2) = 0 \\ \frac{\partial \mathcal{L}}{\partial n_t(i)} &= -\eta \frac{1}{n_t(i)} + \lambda_t W_t(i) = 0, \qquad i = 1, 2. \end{aligned}$$

In addition

$$\frac{\partial \mathcal{L}}{\partial c_t} = \frac{1}{c_t} - \lambda_t P_t = 0.$$

This gives

$$\frac{P_t(1)c_t(1)}{P_t(2)c_t(2)} = \frac{\phi}{1-\phi}.$$

From the consumption index

$$\begin{aligned}
\frac{c_t(1)}{c_t} &= \phi^\phi(1-\phi)^{1-\phi}\left(\frac{c_t(1)}{c_t(2)}\right)^{1-\phi} = \phi^\phi(1-\phi)^{1-\phi}\left(\frac{\phi}{1-\phi}\frac{P_t(2)}{P_t(1)}\right)^{1-\phi} \\
&= \phi\left(\frac{P_t(2)}{P_t(1)}\right)^{1-\phi} \\
\frac{c_t(2)}{c_t} &= (1-\phi)\left(\frac{P_t(1)}{P_t(2)}\right)^{\phi}.
\end{aligned}$$

Hence

$$\begin{aligned}
P_t &= P_t(1)\frac{c_t(1)}{c_t} + P_t(2)\frac{c_t(2)}{c_t} \\
&= P_t(1)^\phi P_t(2)^{1-\phi}.
\end{aligned}$$

The levels of employment satisfy

$$\begin{aligned}
\frac{W_t(1)n_t(1)}{P_t(1)c_t(1)} &= \frac{\eta}{\phi} \\
\frac{W_t(2)n_t(2)}{P_t(2)c_t(2)} &= \frac{\eta}{1-\phi} \\
\frac{W_t(i)n_t(i)}{P_t c_t} &= \eta, \qquad 1 = 1, 2.
\end{aligned}$$

If labour markets are competitive then $W_t(i) = W_t$, the common wage rate. Hence, total employment is

$$n_t = n_t(1) + n_t(2) = 2\eta\frac{P_t c_t}{W_t}.$$

(b) Each firm maximizes its profits subject to the demand for its product as given by the relations $\frac{c_t(i)}{c_t}$ above. Because $c_t(i) = y_t(i) = A_{it}n_t(i)$, the first-order condition of $\Pi_t(i) = P_t(i)y_t(i) - W_t(i)n_t(i)$ with respect to $c_t(i)$ is

$$\frac{d\Pi_t(i)}{dc_t(i)} = P_t(i) + \frac{\partial P_t(i)}{\partial c_t(i)}c_t(i) - W_t\frac{dn_t(i)}{dc_t(i)} = 0.$$

As

$$
\begin{aligned}
\frac{dc_t(i)}{dn_t(i)} &= \frac{dy_t(i)}{dn_t(i)} = A_{it} \\
\frac{\partial c_t(1)}{\partial P_t(1)} &= -(1-\phi)\frac{c_t(1)}{P_t(1)} \\
\frac{\partial c_t(2)}{\partial P_t(2)} &= -\phi\frac{c_t(2)}{P_t(2)}
\end{aligned}
$$

we obtain

$$
\begin{aligned}
P_t(1) &= (1-\phi)\frac{W_t}{A_{1t}} \\
P_t(2) &= \phi\frac{W_t}{A_{2t}}.
\end{aligned}
$$

Hence an increase in W_t has different effects on firm prices due to a demand factor - their elasticity in the consumption index - and a supply factor - the productivity of labor in each firm.

9.3. Consider a model with two intermediate goods where final output is related to intermediate inputs through

$$
y_t = \frac{y_t(1)^{\phi} y_t(2)^{1-\phi}}{\phi^{\phi}(1-\phi)^{1-\phi}}
$$

and the final output producer chooses the inputs $y_t\,(1)$ and $y_t(2)$ to maximize the profits of the final producer

$$
\Pi_t = P_t y_t - P_t(1) y_t(1) - P_t(2) y_t(2)
$$

where P_t is the price of final output and $P_t(i)$ are the prices of the intermediate inputs. Intermediate goods are produced with the production function

$$
y_t(i) = A_{it} n_t(i)^{\alpha}
$$

where $n_t(i)$ is labour input and the intermediate goods firms maximize the profit function

$$
\Pi_t(i) = P_t(i) y_t(i) - W_t n_t(i)
$$

where W_t is the economy-wide wage rate.

(a) Derive the demand functions for the intermediate inputs.

(b) Derive their supply functions.

(c) Hence examine whether there is an efficiency loss for total output.

Solution

(a) The demand functions for the intermediate goods are derived from the decisions of the final producer. Profits for the final producer are therefore

$$\Pi_t = P_t \frac{y_t(1)^{\phi} y_t(2)^{1-\phi}}{\phi^{\phi}(1-\phi)^{1-\phi}} - P_t(1)y_t(1) - P_t(2)y_t(2).$$

The first-order conditions for maximizing profits are

$$\begin{aligned}
\frac{\partial \Pi_t}{\partial y_t(1)} &= P_t \phi \frac{y_t}{y_t(1)} - P_t(1) = 0 \\
\frac{\partial \Pi_t}{\partial y_t(2)} &= P_t(1-\phi) \frac{y_t}{y_t(2)} - P_t(2) = 0.
\end{aligned}$$

Hence the demands for the inputs are

$$\begin{aligned}
y_t(1) &= \phi \frac{P_t}{P_t(1)} y_t \\
y_t(2) &= (1-\phi) \frac{P_t}{P_t(2)} y_t
\end{aligned}$$

and

$$P_t = P_t(1)^{\phi} P_t(2)^{1-\phi}.$$

(b) The supply of intermediate goods is obtained from maximizing the profits of the intermediate goods firms taking the demand for their outputs as derived above. For the first intermediate good producer profits can be written as

$$\begin{aligned}
\Pi_t(1) &= \phi P_t y_t - W_t n_t(1) \\
&= \phi P_t A_{1t} n_t(1)^{\alpha_1} - W_t n_t(1).
\end{aligned}$$

Maximising $\Pi_t(1)$ with respect to $n_t(1)$, taking P_t and y_t as given, yields

$$\begin{aligned} n_t(1) &= \phi \frac{P_t}{W_t} y_t \\ y_t(1) &= A_{1t}[\phi \frac{P_t}{W_t} y_t]^\alpha \\ P_t(1) &= A_{1t}^{-\frac{1}{\alpha}} y_t(1)^{\frac{1-\alpha}{\alpha}} W_t. \end{aligned}$$

Similarly for good two

$$\begin{aligned} n_t(2) &= (1-\phi) \frac{P_t}{W_t} y_t \\ y_t(2) &= A_{2t}[(1-\phi) \frac{P_t}{W_t} y_t]^\alpha \\ P_t(2) &= A_{1t}^{-\frac{1}{\alpha}} y_t(2)^{\frac{1-\alpha}{\alpha}} W_t. \end{aligned}$$

(c) Total output is derived from the outputs of the intermediate goods as

$$\begin{aligned} y_t(1) &= A_{1t} n_t(1)^\alpha = \phi \frac{P_t}{P_t(1)} y_t \\ y_t(2) &= A_{2t} n_t(2)^\alpha = (1-\phi) \frac{P_t}{P_t(2)} y_t. \end{aligned}$$

Total employment is

$$\begin{aligned} n_t &= n_t(1) + n_t(2) \\ &= \left(\frac{\phi P_t y_t}{A_{1t} P_t(1)} \right)^{\frac{1}{\alpha}} + \left(\frac{(1-\phi) P_t y_t}{A_{2t} P_t(2)} \right)^{\frac{1}{\alpha}}. \end{aligned}$$

This gives a relation between the output of the final good and aggregate employment which can be written as

$$\begin{aligned} y_t &= v_t n_t^\alpha \\ v_t &= \left[\left(\frac{\phi}{A_{1t} P_t(1)} \right)^{\frac{1}{\alpha}} + \left(\frac{1-\phi}{A_{2t} P_t(2)} \right)^{\frac{1}{\alpha}} \right]^{-\alpha} (P_t y_t)^{-1}. \end{aligned}$$

If $v_t < 1$ then there is an efficiency loss in the use of labor in producing the final output as compared with intermediate outputs.

9.4. Consider pricing with intermediate inputs where the demand for an intermediate firm's output is

$$y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\phi} y_t$$

its profit is

$$\Pi_t(i) = P_t(i)y_t(i) - C_t(i)$$

and its total cost is

$$C_t(i) = \frac{\phi}{1-\phi}\ln[P_t(i)y_t(i)].$$

(a) Find the optimal price $P_t(i)^*$ if the firm maximizes profits period by period while taking y_t and P_t as given.

(b) If instead the firm chooses a price which it plans to keep constant for all future periods and hence maximizes $\Sigma_{s=0}^{\infty}(1+r)^{-s}\Pi_{t+s}(i)$, derive the resulting optimal price $P_t(i)^{\#}$.

(c) What is this price if expressed in terms of $P_t(i)^*$?

(d) Hence comment on the effect on today's price of anticipated future shocks to demand and costs.

Solution

(a) The profit function is

$$\Pi_t(i) = P_t(i)\left(\frac{P_t(i)}{P_t}\right)^{-\phi} y_t - \frac{\phi}{1-\phi}\ln[P_t(i)\left(\frac{P_t(i)}{P_t}\right)^{-\phi} y_t]$$

The first-order condition for maximizing profits is

$$\frac{\partial \Pi_t(i)}{\partial P_t(i)} = (1-\phi)\left(\frac{P_t(i)}{P_t}\right)^{-\phi} y_t - \phi\frac{1}{P_t(i)} = 0$$

Hence the optimal price is

$$P_t^*(i) = \left(\frac{\phi}{1-\phi}P_t^{-\phi}y_t^{-1}\right)^{\frac{1}{1-\phi}}.$$

(b) Maximizing V_t, the present value of profits, and keeping $P_t(i)$, the prices of intermediate goods, constant for all future periods, implies that

$$\begin{aligned} V_t &= \Sigma_{s=0}^{\infty}(1+r)^{-s}\Pi_{t+s}(i) \\ &= \Sigma_{s=0}^{\infty}(1+r)^{-s}[P_t(i)\left(\frac{P_t(i)}{P_{t+s}}\right)^{-\phi}y_t - \frac{\phi}{1-\phi}\ln[P_t(i)\left(\frac{P_t(i)}{P_{t+s}}\right)^{-\phi}y_{t+s}]. \end{aligned}$$

The first-order condition is

$$\frac{\partial V_t}{\partial P_t(i)} = \Sigma_{s=0}^{\infty}(1+r)^{-s}[(1-\phi)\left(\frac{P_t(i)}{P_{t+s}}\right)^{-\phi}y_{t+s} - \phi\frac{1}{P_t(i)}] = 0.$$

Hence, the optimal price is

$$P_t^{\#}(i) = \left(\frac{\phi}{1-\phi}\right)^{\frac{1}{1-\phi}}\left[\frac{1}{r}\Sigma_{s=0}^{\infty}(1+r)^{-(s+1)}P_{t+s}{}^{\phi}y_{t+s}\right]^{-\frac{1}{1-\phi}}.$$

(c) Expressed in terms of $P_t^*(i)$ we obtain

$$P_t^{\#}(i) = \left[\frac{1}{r}\Sigma_{s=0}^{\infty}(1+r)^{-(s+1)}P_{t+s}^*(i)^{-(1-\phi)}\right]^{-\frac{1}{1-\phi}}.$$

(d) It follows that anticipated future shocks to demand and costs will affect future prices $P_{t+s}^*(i)$ $(s>0)$ and hence the current price $P_t^{\#}(i)$.

9.5. Consider an economy with two sectors $i = 1, 2$. Each sector sets its price for two periods but does so in alternate periods. The general price level in the economy is the average of sector prices: $p_t = \frac{1}{2}(p_{1t} + p_{2t})$, hence $p_t = \frac{1}{2}(p_{it}^{\#} + p_{i+1,t-1}^{\#})$. In the period the price is reset it is determined by the average of the current and the expected future optimal price: $p_{it}^{\#} = \frac{1}{2}(p_{it}^* + E_t p_{i,t+1}^*)$, $i = 1, 2$. The optimal price is assumed to be determined by $p_{it}^* - p_t = \phi(w_t - p_t)$, where w_t is the wage rate.

(a) Derive the general price level if wages are generated by $\Delta w_t = e_t$, where e_t is a zero mean *i.i.d.* process. Show that p_t can be given a forward-looking, a backward-looking and a univariate representation.

(b) If the price level in steady state is p, how does the price level in period t respond to an unanticipated shock in wages in period t?

(c) How does the price level deviate from p in period t in response to an anticipated wage shock in period $t+1$?

Solution

(a) It follows that

$$\begin{aligned} p_{it}^{\#} &= \frac{1}{2}[(1-\phi)(p_t + E_t p_{t+1}) + \phi(w_t + E_t w_{t+1})] \\ p_{i+1,t-1}^{\#} &= \frac{1}{2}[(1-\phi)(p_{t-1} + E_{t-1} p_t) + \phi(w_{t-1} + E_{t-1} w_t)] \end{aligned}$$

hence

$$p_t = \frac{1}{4}\left[(1-\phi)(p_{t-1} + E_{t-1}p_t + p_t + E_t p_{t+1}) + \phi(w_{t-1} + E_{t-1}w_t + w_t + E_t w_{t+1})\right].$$

Let $\varepsilon_t = p_t - E_{t-1}p_t$ and $e_t = w_t - E_{t-1}w_t$ then

$$p_t = \frac{1}{4}\left[(1-\phi)(p_{t-1} + p_t - \varepsilon_t + p_t + E_t p_{t+1}) + \phi(w_{t-1} + w_t - e_t + w_t + E_t w_{t+1})\right].$$

Or, introducing the lag operator $Lx_t = x_{t-1}$ and $L^{-1} = E_t x_{t+1}$,

$$[1 - \frac{1}{2}(1-\phi) - \frac{1}{4}(1-\phi)(L + L^{-1})]p_t = -\frac{1}{4}(1-\phi)\varepsilon_t + [\frac{1}{2}\phi + \frac{1}{4}\phi(L + L^{-1})]w_t - \frac{1}{4}\phi e_t.$$

If $1 > \phi > \frac{1}{3}$, then

$$1 - \frac{1}{2}(1-\phi) - \frac{1}{4}(1-\phi)(L + L^{-1}) = -\frac{1}{4}(1-\phi)(L - \lambda_1)(L - \lambda_2)L^{-1},$$

where

$$(L-\lambda_1)(L-\lambda_2)|_{L=1} = -\frac{[1 - \frac{1}{2}(1-\phi) - \frac{1}{4}(1-\phi)(L + L^{-1})]L}{\frac{1}{4}(1-\phi)}|_{L=1} = -\frac{\phi}{\frac{1}{4}(1-\phi)} < 0.$$

Hence we have a unique saddlepath solution. If we let $\lambda_1 > 1$ and $\lambda_2 < 1$ then

$$\frac{1}{4}(1-\phi)\lambda_1(1 - \lambda_1^{-1}L)(1 - \lambda_2 L^{-1})p_t = -\frac{1}{4}(1-\phi)\varepsilon_t + [\frac{1}{2}\phi + \frac{1}{4}\phi(L + L^{-1})]w_t - \frac{1}{4}\phi e_t,$$

or

$$p_t = \frac{1}{\lambda_1} p_{t-1} - \frac{1}{\lambda_1}\varepsilon_t - \frac{\phi}{\lambda_1(1-\phi)}[w_{t-1} - w_t + e_t + (2 + \lambda_2 + \lambda_2^{-1})\Sigma_{s=0}^{\infty}\lambda_2^s E_t w_{t+s}].$$

Thus, in general, p_t is forward-looking.

p_t can, however, be given a backward-looking representation using the fact that $\Delta w_t = e_t$. This implies that $E_t w_{t+s} = w_t$ and therefore

$$p_t = \frac{1}{\lambda_1} p_{t-1} - \frac{1}{\lambda_1}\varepsilon_t - \frac{\phi(2+\lambda_2+\lambda_2^{-1})}{\lambda_1(1-\phi)(1-\lambda_2)} w_t.$$

It follows that

$$\begin{aligned}
\varepsilon_t &= p_t - E_{t-1}p_t \\
&= -\frac{1}{\lambda_1}\varepsilon_t - \frac{\phi(2+\lambda_2+\lambda_2^{-1})}{\lambda_1(1-\phi)(1-\lambda_2)} e_t \\
&= -\frac{\phi(2+\lambda_2+\lambda_2^{-1})}{(1+\lambda_1)(1-\phi)(1-\lambda_2)} e_t.
\end{aligned}$$

Hence the price level is generated by the process

$$p_t = \frac{1}{\lambda_1} p_{t-1} - \frac{\phi(2+\lambda_2+\lambda_2^{-1})}{\lambda_1(1-\phi)(1-\lambda_2)}[w_t + \frac{\lambda_1}{1+\lambda_1} e_t].$$

p_t can also be given a univariate representation. As $\Delta w_t = e_t$, p_t can be re-written as the ARIMA(1,1,1) process

$$\Delta p_t = \frac{1}{\lambda_1}\Delta p_{t-1} - \frac{\phi(2+\lambda_2+\lambda_2^{-1})}{\lambda_1(1-\phi)(1-\lambda_2)}[e_t + \frac{\lambda_1}{1+\lambda_1}\Delta e_t]$$

(b) From $w_t = w_{t-1} + e_t$, an unanticipated shock e_t in period t causes the price in period t to be

$$p_t = p - \frac{\phi(2+\lambda_2+\lambda_2^{-1})}{\lambda_1(1-\phi)(1-\lambda_2)}(1 + \frac{\lambda_1}{1+\lambda_1})e_t.$$

The last term gives the required effect.

(c) From $w_{t+1} = w_t + e_{t+1}$, where e_{t+1} is known in period t,

$$\begin{aligned}
p_t &= \frac{1}{\lambda_1}p_{t-1} - \frac{1}{\lambda_1}\varepsilon_t - \frac{\phi}{\lambda_1(1-\phi)}[w_{t-1} - w_t + e_t + (2+\lambda_2+\lambda_2^{-1})\Sigma_{s=0}^{\infty}\lambda_2^s E_t w_{t+s}] \\
&= \frac{1}{\lambda_1}p_{t-1} - \frac{\phi(2+\lambda_2+\lambda_2^{-1})}{\lambda_1(1-\phi)(1-\lambda_2)}[w_t + \frac{\lambda_1}{1+\lambda_1}e_t + \lambda_2 e_{t+1}] \\
&= p - \frac{\phi(2+\lambda_2+\lambda_2^{-1})\lambda_2}{\lambda_1(1-\phi)}e_{t+1}
\end{aligned}$$

Again the last term gives the required effect.

Chapter 10

10.1. (a) Suppose that a consumer's initial wealth is given by W_0, and the consumer has the option of investing in a risky asset which has a rate of return r or a risk-free asset which has a sure rate of return f. If the consumer maximizes the expected value of a strictly increasing, concave utility function $U(W)$ by choosing to hold the risky versus the risk-free asset, and if the variance of the return on the risky asset is $V(r)$, find an expression for the risk premium ρ that makes the consumer indifferent between holding the risky and the risk-free asset.

(b) Explain how absolute risk aversion differs from relative risk aversion.

(c) Suppose that the consumer's utility function is the hyperbolic absolute risk aversion (HARA) function

$$U(W) = \frac{1-\sigma}{\sigma}\left[\frac{\alpha W}{1-\sigma} + \beta\right]^{\sigma}, \quad \alpha > 0,\ \beta > 0, ; 0 < \sigma < 1.$$

Discuss how the magnitude of the risk premium varies as a function of wealth and the parameters α, β, and σ.

Solution

(a) If the investor holds the risk-free asset then

$$EU(W) = U[W_o(1+f)].$$

If the investor holds the risky asset then for the investor to be indifferent between holding the risky and risk-free asset requires the risk premium ρ to satisfy

$$EU[W] = EU[W_0(1+r)] = U[W_o(1+f)].$$

Expanding $EU[W_0(1+r)]$ about $r = f + \rho$ we obtain

$$E[U(W)] \simeq U[W_0(1+f+\rho)] + \frac{1}{2}W_0^2 E(r-f-\rho)^2 U''.$$

Expanding $U[W_0(1+f+\rho)]$ about $\rho = 0$ we obtain

$$U[W_0(1+f+\rho)] \simeq U[W_0(1+f)] + W_0\rho U'.$$

Noting that $E(r-f-\rho)^2 \simeq V(r)$,

$$E[U(W)] \simeq U[W_o(1+f)] + W_0\rho U' + W_0^2\frac{V(r)}{2}U''.$$

Hence the risk premium is approximately

$$\rho \simeq -\frac{V(r)}{2}\frac{W_0U''}{U'}.$$

where $-\frac{W_0U''}{U'}$ is the coefficient of relative risk aversion.

(b) The coefficient of absolute risk aversion is defined by $-\frac{U''}{U'}$ whereas the coefficient of relative risk aversion (CRRA) is $-\frac{WU''}{U'}$ if utility is a function of wealth - or $-\frac{cU''}{U'}$ if utility is a function of consumption. In general, whereas the CRRA is independent of wealth the CARA is not.

(c) For the HARA utility function

$$\begin{aligned} U' &= \frac{U}{\frac{W}{1-\sigma}+\frac{\beta}{\alpha}} \\ U'' &= -\frac{U}{[\frac{W}{1-\sigma}+\frac{\beta}{\alpha}]^2} \\ CRRA &= -\frac{W_0U''}{U'} = [\frac{W_0}{1-\sigma}+\frac{\beta}{\alpha}]^{-1}. \end{aligned}$$

This shows that the CRRA depends on wealth. It follows that the risk premium is

$$\begin{aligned} \rho &\simeq -\frac{V(r)}{2}\frac{W_0U''}{U'} \\ &= \frac{V(r)}{2}\frac{W_0}{\frac{W_0}{1-\sigma}+\frac{\beta}{\alpha}}. \end{aligned}$$

The risk premium increases the larger is W_0 and α and the smaller are β and σ. The HARA utility function therefore implies that the risk premium depends on wealth. Power utility, which has a constant CRRA, does not depend on wealth.

10.2. Consider there exists a representative risk-averse investor who derives utility from current and future consumption according to

$$\mathcal{U} = \Sigma_{s=0}^{\theta}\beta^s E_t U(c_{t+s}),$$

where $0 < \beta < 1$ is the consumer's subjective discount factor, and the single-period utility function has the form

$$U(c_t) = \frac{c_t^{1-\sigma} - 1}{1 - \sigma}, \quad \sigma \geq 0.$$

The investor receives a random exogenous income of y_t and can save by purchasing shares in a stock or by holding a risk-free one-period bond with a face-value of unity. The ex-dividend price of the stock is given by P_t^S in period t. The stock pays a random stream of dividends D_{t+s} per share held at the end of the previous period. The bond sells for P_t^B in period t.

(a) Find an expression for the bond price that must hold at the investor's optimum.

(b) Find an expression for the stock price that must hold at the investor's optimum. Interpret this expression.

(c) Derive an expression for the risk premium on the stock that must hold at the investor's optimum. Interpret this expression.

Solution

(a) The problem can be re-written as involving the recursion

$$\mathcal{U}_t = U(c_t) + \beta E_t \mathcal{U}_{t+1}$$

and solved using stochastic dynamic programming. Let the number of shares held during period t be N_t^S and the number of bonds be N_t^B. At the start of each period the investor can sell the stocks held over the previous period as well as receiving the dividends on these shares. The investor's nominal budget constraint can therefore be expressed as

$$P_t^S N_t^S + P_t^B N_t^B + P_t c_t = P_t y_t + (P_t^S + D_t) N_{t-1}^S + N_{t-1}^B$$

where P_t is the general price level.

Maximizing $\mathcal{U}_t$ with respect to $\{c_{t+s}, N_{t+s}^S, N_{t+s}^B\}$ subject to the budget constraint gives the following first-order condition

$$\frac{\partial \mathcal{U}_t}{\partial c_t} = \frac{\partial U_t}{\partial c_t} + \beta E_t \left(\frac{\partial \mathcal{U}_{t+1}}{\partial c_t}\right) = 0$$

where

$$\begin{aligned}
\frac{\partial \mathcal{U}_{t+1}}{\partial c_t} &= \frac{\partial \mathcal{U}_{t+1}}{\partial c_{t+1}} \frac{\partial c_{t+1}}{\partial c_t} \\
&= \frac{\partial U_{t+1}}{\partial c_{t+1}} \frac{\partial c_{t+1}}{\partial c_t} \\
&= -c_{t+1}^{-\sigma} \frac{P_t}{P_t^B P_{t+1}}
\end{aligned}$$

and $\frac{\partial c_{t+1}}{\partial c_t}$ is obtained from the budget constraints for periods t and $t+1$ as

$$\frac{\partial c_{t+1}}{\partial c_t} = \frac{\partial N_t^B}{\partial c_t} \frac{\partial c_{t+1}}{\partial N_t^B} = -\frac{P_t}{P_t^B} \frac{1}{P_{t+1}}.$$

Hence,

$$\frac{\partial \mathcal{U}_t}{\partial c_t} = c_t^{-\sigma} - \beta E_t[c_{t+1}^{-\sigma} \frac{P_t}{P_t^B P_{t+1}}] = 0.$$

The consumption Euler equation is therefore

$$E_t \left[\beta (\frac{c_{t+1}}{c_t})^{-\sigma} \frac{P_t}{P_t^B P_{t+1}} \right] = 1.$$

It follows that the bond price can be expressed as

$$P_t^B = E_t \left[\beta (\frac{c_{t+1}}{c_t})^{-\sigma} \frac{P_t}{P_{t+1}} \right].$$

The expression $\beta(\frac{c_{t+1}}{c_t})^{-\sigma} \frac{P_t}{P_{t+1}}$ is known as the pricing kernel.

(b) Alternatively we may obtain $\frac{\partial c_{t+1}}{\partial c_t}$ from

$$\frac{\partial c_{t+1}}{\partial c_t} = \frac{\partial N_t^S}{\partial c_t} \frac{\partial c_{t+1}}{\partial N_t^S} = -\frac{P_t}{P_t^S} \frac{P_{t+1}^S + D_{t+1}}{P_{t+1}},$$

It then follows that the Euler equation can be written as

$$E_t \left[\beta (\frac{c_{t+1}}{c_t})^{-\sigma} \frac{P_t (P_{t+1}^S + D_{t+1})}{P_t^S P_{t+1}} \right] = 1.$$

To evaluate this we re-write it as

$$E_t \left[\beta (\frac{c_{t+1}}{c_t})^{-\sigma} \frac{P_t}{P_{t+1}} \cdot \frac{P_{t+1}^S + D_{t+1}}{P_t^S} \right] = 1.$$

As

$$E_t[X_{t+1}Y_{t+1}] = E_t[X_{t+1}]E_t[Y_{t+1}] + Cov_t(X_{t+1}, Y_{t+1})$$

we obtain

$$\begin{aligned} 1 &= E_t\left[\beta(\frac{c_{t+1}}{c_t})^{-\sigma}\frac{P_t}{P_{t+1}}\right]E_t\left[\frac{P^S_{t+1}+D_{t+1}}{P^S_t}\right] \\ & \quad -Cov_t[\beta(\frac{c_{t+1}}{c_t})^{-\sigma}\frac{P_t}{P_{t+1}}, \frac{P^S_{t+1}+D_{t+1}}{P^S_t}]. \end{aligned}$$

From the previous expression for P^B_t, it follows that

$$E_t\left[\frac{P^S_{t+1}+D_{t+1}}{P^S_t}\right] = \frac{1 - Cov_t[\beta(\frac{c_{t+1}}{c_t})^{-\sigma}\frac{P_t}{P_{t+1}}, \frac{P^S_{t+1}+D_{t+1}}{P^S_t}]}{P^B_t}.$$

Denoting the right-hand side by z_t, the stock price therefore evolves according to the forward-looking equation

$$\begin{aligned} P^S_t &= \frac{1}{z_t}E_t(P^S_{t+1}+D_{t+1}) \\ &= E_t\Sigma_{i=1}\frac{D_{t+i}}{\Pi^{i-1}_{j=0}z_{t+j}}. \end{aligned}$$

Thus the stock price depends on the expected discounted value of future dividends.

(c) We may interpret $\frac{P^S_{t+1}+D_{t+1}}{P^S_t} = 1 + R^S_{t+1}$, where R^S_{t+1} is the nominal equity return, and $\frac{1}{P^B_t} = 1+R^B_t$, where R^B_t is the risk-free return on bonds. Hence we can re-write the price equation for stocks as

$$\begin{aligned} E_t\left[\frac{P^S_{t+1}+D_{t+1}}{P^S_t}\right] &= E_t(1+R^S_{t+1}) \\ &= 1+R^B_t - (1+R^B_t)Cov_t[\beta(\frac{c_{t+1}}{c_t})^{-\sigma}\frac{P_t}{P_{t+1}}, \frac{P^S_{t+1}+D_{t+1}}{P^S_t}]. \end{aligned}$$

This gives

$$E(R^S_{t+1} - R^B_t) = -(1+R^B_t)Cov_t[\beta(\frac{c_{t+1}}{c_t})^{-\sigma}\frac{P_t}{P_{t+1}}, \frac{P^S_{t+1}+D_{t+1}}{P^S_t}]$$

where the right-hand side is the equity risk premium. It shows that risk arises due to conditional (and time-varying) covariation between the rate of growth of consumption and the real return to equity.

10.3. If the pricing kernel is M_{t+1}, the return on a risky asset is r_t and that on a risk-free asset is f_t,

(a) state the asset-pricing equation for the risky asset and the associated risk premium.

(b) Express the risk premium as a function of the conditional variance of the risky asset and give a regression interpretation of your result.

Solution

(a) As shown in Chapter 10 the asset-pricing equation is

$$E_t \left[M_{t+1}(1 + r_{t+1}) \right] = 1$$

and the risk premium is given by

$$E_t(r_{t+1} - f_t) = (1 + f_t) Cov_t(M_{t+1}, r_{t+1}).$$

(b) We can re-write the risk premium as

$$\begin{aligned} E_t(r_{t+1} - f_t) &= (1 + f_t) \frac{Cov_t(M_{t+1}, r_{t+1})}{V_t(r_{t+1})} \cdot V_t(r_{t+1}) \\ &= (1 + f_t) b_t V_t(r_{t+1}) \end{aligned}$$

where

$$b_t = \frac{Cov_t(M_{t+1}, r_{t+1})}{V_t(r_{t+1})}.$$

b_t can be interpreted as the conditional regression coefficient of M_{t+1} on r_{t+1}. $V_t(r_{t+1})$ can be interpreted as the quantity of risk and $(1 + f_t)b_t$ as the price of risk. Conditional terms like these appear naturally in up-dating formulae such as in the Kalman filter, or in recursive or rolling regressions. In general equilibrium M_{t+1} is a function of consumption. b_t can therefore be interpreted as expressing risk in terms of consumption.

10.4. (a) What is the significance of an asset having the same pay-off in all states of the world?

(b) Consider a situation involving three assets and two states. Suppose that one asset is a risk-free bond with a return of 20%, a second asset has a price of 100 and pay-offs of 60 and 200 in the two states, and a third asset has pay-offs of 100 and 0 in the two states. If the probability of the first state occurring is 0.4.

(i) What types of assets might this description fit?

(ii) Find the prices of the implied contingent claims in the two states.

(iii) Find the price of the third asset.

(iv) What is the risk premium associated with the second asset?

Solution

(a) An asset that has the same pay-off in all states of the world is a risk-free asset.

(b) (i) This description fits the situation where we have the following three types of asset:

$$\begin{aligned} B(t) &= \text{riskless borrowing and lending} \\ S(t) &= \text{a stock price} \\ C(t) &= \text{the value of a call option with strike price } K \end{aligned}$$

where a call option gives the owner the right to purchase the underlying asset at the strike price (exercise price) K. Suppose that there are possible states of nature in period $t+1$. If we let $X(t)$ denote the vector of pay-offs on the different assets in the future states and $P(t)$ denote a vector of asset prices

$$P(t) = \begin{bmatrix} B(t) \\ S(t) \\ C(t) \end{bmatrix}.$$

If f denotes the riskless rate of interest then the pay-off matrix is

$$X(t) = \begin{bmatrix} (1+f)B(t) & (1+f)B(t) \\ S_1(t+1) & S_2(t+1) \\ C_1(t+1) & C_2(t+1) \end{bmatrix}.$$

where we can let $B(t) = 1$ as a normalization. It then follows that

$$P(t) = X(t)q(t)$$

where $q(t)$ is a vector of state contingent claims prices.

In the present case we have three assets and two states giving

$$\begin{bmatrix} 1 \\ 100 \\ C \end{bmatrix} = \begin{bmatrix} 1.2 & 1.2 \\ 60 & 200 \\ 0 & 100 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix},$$

where in effect $K = 100$.

(ii) We can determine the two state prices q_1 and q_2 from the first two equations to obtain $q_1 = 0.060$ and $q_2 = 0.357$.

(iii) We can then use the third equation to determine the value of the third asset (the option) as $C(t) = 3.57$.

(iv) The risk premium $\rho^S(t)$ associated with the second asset is

$$\rho^S(t) = \frac{E[S(t+1)]}{S(t)} - (1+f)$$

where, denoting the probability of state s occurring by $\pi(s)$,

$$\begin{aligned} E[S(t+1)] &= \pi(1)S_1(t+1) + \pi(2)S_2(t+1) \\ &= 0.4 \times 60 + 0.6 \times 200 = 144. \end{aligned}$$

As the risk-free rate is 0.2 and $S(t) = 100$, the risk premium is

$$\begin{aligned} \rho^S(t) &= \frac{144}{100} - 1.2 \\ &= 0.24. \end{aligned}$$

A risk premium for the call option can be calculated in the same way and is $\frac{0.6 \times 100}{3.57} - 1.2 = 15.6$.

10.5. Consider the following two-period problem for a household in which there is one state of the world in the first period and two states in the second period. Income in the first period is 6; in the second period it is 5 in state one which occurs with probability 0.2, and is 10 in state two. There is a risk-free bond with a rate of return equal to 0.2. If instantaneous utility is $\ln c_t$ and the rate of time discount is 0.2 find

(a) the levels of consumption in each state,

(b) the state prices,

(c) the stochastic discount factors,

(d) the risk-free "rate of return" (i.e. rate of change) to income in period two,

(e) the "risk premium" for income in period two.

Solution

(a) The problem is to maximize

$$V = U(c) + \beta\{\pi(1)U[c(1)] + \pi(2)U[c(2)]\}$$

subject to the intertempral budget constraint

$$c + q(1)c(1) + q(2)c(2) = y + q(1)y(1) + q(2)y(2)$$

where $q(1)$ and $q(2)$ are the prices of contingent claims in the two states in the second period. The Lagrangian is

$$\begin{aligned} \mathcal{L} &= U(c) + \beta\{\pi(1)U[c(1)] + \pi(2)U[c(2)]\} \\ &\quad +\lambda\left[y + q(1)y(1) + q(2)y(2)y - c - q(1)c(1) - q(2)c(2)\right]. \end{aligned}$$

The first-order conditions are given by

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c} &= U'(c) - \lambda = 0, \\ \frac{\partial \mathcal{L}}{\partial c(s)} &= \beta\pi(s)U'[c(s)] - \lambda q(s) = 0, \quad s = 1, 2. \end{aligned}$$

Thus

$$q(s) = \beta\pi(s)\frac{U'[c(s)]}{U'(c)}, \quad s = 1, 2.$$

Hence any income stream $x(s)$ can be priced using

$$p = \Sigma_s q(s)x(s) = \frac{\beta\Sigma_s\pi(s)U'[c(s)]x(s)}{U'(c)}.$$

In particular, we can price the income stream $y(s)$ by setting $x(s) = y(s)$. And if there is a risk-free asset then its price given a unit pay-off is

$$\Sigma_s q(s) = \Sigma_s\pi(s)\frac{\beta U'[c(s)]}{U'(c)} = \frac{1}{1+f}$$

where f is the risk-free rate of return.

Using the given values we obtain

$$\begin{aligned} q(1) &= \beta\pi(1)\frac{c}{c(1)} = \frac{1}{1.2}0.2\frac{c}{c(1)} \\ q(2) &= \beta\pi(2)\frac{c}{c(2)} = \frac{1}{1.2}0.8\frac{c}{c(2)}. \end{aligned}$$

We now evaluate the budget constraint noting that

$$c + q(1)c(1) + q(2)c(2) = c + \frac{0.2}{1.2}c + \frac{0.8}{1.2}c = \frac{2.2}{1.2}c$$

and that

$$y + q(1)y(1) + q(2)y(2) = 2y = 12.$$

Therefore

$$c = \frac{1.2 \times 12}{2.2} = 6.55.$$

From the price of income

$$\frac{1}{1.2}0.2\frac{c}{c(1)}5 + \frac{1}{1.2}0.8\frac{c}{c(2)}10 = 6$$

or

$$\frac{5c}{c(1)} + \frac{40c}{c(2)} = 36$$

From the risk-free bond

$$\frac{1}{1.2}0.2\frac{c}{c(1)} + \frac{1}{1.2}0.8\frac{c}{c(2)} = \frac{1}{1.2}$$

or

$$\frac{c}{c(1)} + \frac{4c}{c(2)} = 5.$$

Solving for $c(1)$ and $c(2)$ from the two equations gives $c(1) = \frac{14}{5}c = 18.3$ and $c(2) = \frac{20}{11}c = 11.9$.

(b) The contingent state prices are $q(1) = \frac{0.2}{1.2}\frac{c}{c(1)} = 0.06$ and $q(2) = \frac{0.8}{1.2}\frac{c}{c(2)} = 0.37$.

(c) The stochastic discount factors are

$$m(s) = \frac{q(s)}{\pi(s)} = \frac{\beta U'[c(s)]}{U'(c)}.$$

Thus, $m(1) = 0.30$ and $m(2) = 0.46$.

(d) If the rate of return to income is $r(s)$ then its expectation satisfies

$$\begin{aligned} E[1 + r(s)] &= \frac{E[y(s)]}{y} = \frac{\pi(1)y(1) + \pi(2)y(2)}{y} \\ &= \frac{0.2 \times 5 + 0.8 \times 10}{6} = 1.5. \end{aligned}$$

Hence $E[r(s)] = 0.5$.

(e) The risk premium for income ρ is therefore

$$\rho = E[r(s) - f] = 0.5 - 0.2 = 0.3.$$

Chapter 11

11.1. An investor with the utility function $U(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$ who maximizes $E_t \Sigma_{s=0}^{\infty} \beta^s U(c_{t+s})$ can either invest in equity with a price of P_t^S and a dividend of D_t or a risk-free one-period bond with nominal return f_t. Derive

(a) the optimal consumption plan, and

(b) the equity premium.

(c) Discuss the effect on the price of equity in period t of a loosening of monetary policy as implemented by an increase in the nominal risk-free rate f_t.

Solution

(a) We start with the Euler equation (see Chapter 10) which is also the general equilibrium asset pricing equation when utility is time separable:

$$E_t \left[\frac{\beta U'_{t+1}(1 + r_{t+1})}{U'_t} \right] = 1$$

where r_{t+1} is a real rate of return and $\beta = \frac{1}{1+\theta}$. In this exercise $U'_t = c_t{}^{-\sigma}$ and the real rate of return depends on the asset. For equity the real return satisfies

$$1 + r_{t+1}^S = \frac{P_{t+1}^S + D_{t+1}}{P_t^S} \frac{P_t}{P_{t+1}} = (1 + R_{t+1}^S) \frac{P_t}{P_{t+1}},$$

where R_{t+1}^S is the nominal return on equity. For bonds

$$1 + r_{t+1}^B = (1 + f_t) \frac{P_t}{P_{t+1}} = \frac{1}{P_t^B} \frac{P_t}{P_{t+1}}$$

where P_t is the general price level and P_t^B is the price of a one-period bond. There is, therefore, no *real* risk-free asset in this problem.

It follows that

$$E_t \left[\beta \left(\frac{c_t}{c_{t+1}} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \cdot (1 + f_t) \right] = 1.$$

As

$$E_t \left[\left(\frac{c_t}{c_{t+1}} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right] = E_t \left[\left(\frac{c_t}{c_{t+1}} \right)^{-\sigma} \right] E_t \left[\frac{P_t}{P_{t+1}} \right] + Cov_t \left[\left(\frac{c_t}{c_{t+1}} \right)^{-\sigma}, \frac{P_t}{P_{t+1}} \right]$$

we obtain the optimal consumption plan

$$\begin{aligned}
E_t\left[\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma}\right] &= \frac{\frac{1+\theta}{1+f_t} - Cov_t\left[\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma}, \frac{P_t}{P_{t+1}}\right]}{E_t\left[\frac{P_t}{P_{t+1}}\right]} \\
&= [E_t(\frac{1}{1+\pi_{t+1}})]^{-1}\left\{\frac{1+\theta}{1+f_t} - Cov_t\left[\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma}, \frac{1}{1+\pi_{t+1}}\right]\right\}
\end{aligned}$$

where $\pi_{t+1} = \frac{\Delta P_{t+1}}{P_t}$ is the inflation rate.

(b) The Euler equation can also be expressed in terms of the return to equity because

$$\begin{aligned}
1 &= E_t\left[\beta\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\cdot(1+R^S_{t+1})\right] \\
&= E_t\left[\beta\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right]E_t\left[1+R^S_{t+1}\right] + Cov_t\left[\beta\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma}\frac{P_t}{P_{t+1}}, (1+R^S_{t+1})\right].
\end{aligned}$$

Solving from the nominal return to equity

$$E_t\left[1+R^S_{t+1}\right] = \frac{1 - Cov_t\left[\beta\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma}\frac{P_t}{P_{t+1}}, R^S_{t+1}\right]}{E_t\left[\beta\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right]}.$$

From part (a)

$$1+f_t = \frac{1}{E_t\left[\beta\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right]}.$$

Hence, subtracting, the expected excess return (the equity premium) is

$$E_t\left[R^S_{t+1} - f_t\right] = \beta(1+f_t)Cov_t\left[\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma}\frac{P_t}{P_{t+1}}, R^S_{t+1}\right].$$

(c) Assuming that a loosening of monetary policy has no significant effect on consumption growth we obtain

$$\begin{aligned}
1+f_t &= \frac{1}{E_t\left[\beta\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right]} \\
&\simeq \frac{1}{E_t\left[\frac{P_t}{P_{t+1}}\right]}.
\end{aligned}$$

Hence inflation is expected to increase. We can approximate the equity pricing equation as

$$E_t\left[R^S_{t+1} - f_t\right] \simeq \beta(1+f_t)Cov_t\left[\frac{P_t}{P_{t+1}}, R^S_{t+1}\right].$$

The nominal return to equity is therefore expected to increase even though this would imply a fall in $Cov_t\left[\frac{P_t}{P_{t+1}}, R^S_{t+1}\right]$. We note that a decrease is not possible because $Cov_t\left[\frac{P_t}{P_{t+1}}, R^S_{t+1}\right]$ would then be positive.

To find the effect on the price of equity note from its return that

$$\begin{aligned} 1 + R^S_{t+1} &= \frac{P^S_{t+1} + D_{t+1}}{P^S_t} \\ &= \frac{P^S_{t+1}}{P^S_t}(1 + \frac{D_{t+1}}{P^S_{t+1}}). \end{aligned}$$

Since inflation has increased we may expect dividends - a nominal variable - to increase too. If the dividend yield is unaffected then this would imply that P^S_{t+1} must increase, and by more than P^S_t. There would then be a capital gain to equity.

11.2. (a) A household with the utility function $U(c_t) = \ln c_t$, which maximizes $E_t\Sigma_{s=0}^{\infty}\beta^s U(c_{t+s})$, can either invest in a one-period domestic risk-free bond with nominal return R_t, or a one-period foreign currency bond with nominal return (in foreign currency) of R^*_t. If the nominal exchange rate (the domestic price of foreign exchange) is S_t derive

(i) the optimal consumption plan, and

(ii) the foreign exchange risk premium.

(b) Suppose that foreign households have an identical utility function but a different discount factor β^*, what is their consumption plan and their risk premium?

(c) Is the market complete? If not,

(i) what would make it complete?

(ii) How would this affect the two risk premia?

Solution

(a) The solution takes a similar form to that in the previous exercise but with the return to the foreign bond replacing the return to equity. Thus,

(i) the optimal consumption plan is

$$\begin{aligned} E_t\left[\frac{c_{t+1}}{c_t}\right] &= \frac{\frac{1+\theta}{1+R_t} - Cov_t\left[\frac{c_{t+1}}{c_t}, \frac{P_t}{P_{t+1}}\right]}{E_t\left[\frac{P_t}{P_{t+1}}\right]} \\ &= [E_t(\frac{1}{1+\pi_{t+1}})]^{-1}\left\{\frac{1+\theta}{1+R_t} - Cov_t\left[\frac{c_{t+1}}{c_t}, \frac{1}{1+\pi_{t+1}}\right]\right\} \end{aligned}$$

where $\pi_{t+1} = \frac{\Delta P_{t+1}}{P_t}$ is the inflation rate.

(ii) The nominal return in domestic currency terms to the foreign bond is $(1+R_t^*)\frac{S_{t+1}}{S_t}$, hence

$$E_t\left[(1+R_t^*)\frac{S_{t+1}}{S_t}\right] = \frac{1 - Cov_t\left[\beta\frac{c_{t+1}}{c_t}\frac{P_t}{P_{t+1}}, (1+R_t^*)\frac{S_{t+1}}{S_t}\right]}{E_t\left[\beta\frac{c_{t+1}}{c_t}\frac{P_t}{P_{t+1}}\right]}.$$

And the foreign exchange risk premium is

$$E_t\left[(1+R_t^*)\frac{S_{t+1}}{S_t} - (1+R_t)\right] = \beta(1+R_t)Cov_t\left[\frac{c_{t+1}}{c_t}\frac{P_t}{P_{t+1}}, (1+R_t^*)\frac{S_{t+1}}{S_t}\right].$$

(b) For the foreign investor the rate of return on foreign bonds expressed in its own currency is $(1+R_t)\frac{S_t}{S_{t+1}}$. Hence the consumption plan is

$$\begin{aligned} E_t\left[\frac{c^*_{t+1}}{c^*_t}\right] &= \frac{\frac{1+\theta^*}{1+R^*_t} - Cov_t\left[\frac{c^*_{t+1}}{c^*_t}, \frac{P^*_t}{P^*_{t+1}}\right]}{E_t\left[\frac{P^*_t}{P^*_{t+1}}\right]} \\ &= [E_t(\frac{1}{1+\pi^*_{t+1}})]^{-1}\left\{\frac{1+\theta^*}{1+R^*_t} - Cov_t\left[\frac{c^*_{t+1}}{c^*_t}, \frac{1}{1+\pi^*_{t+1}}\right]\right\} \end{aligned}$$

and the foreign risk premium is

$$E_t\left[(1+R_t)\frac{S_t}{S_{t+1}} - (1+R_t^*)\right] = \beta^*(1+R_t^*)Cov_t\left[\frac{c^*_{t+1}}{c^*_t}\frac{P^*_t}{P^*_{t+1}}, (1+R_t)\frac{S_t}{S_{t+1}}\right],$$

where c^*_t is foreign consumption and P^*_t is the foreign price level.

(c) The market is not complete as the pricing kernels for the two countries are different due to $\beta \neq \beta^*$.

(i) It would be complete if $\beta = \beta^*$.

(ii) With complete markets consumption would be the same in each country and the exchange rate will be the ratio of the price levels so that $S_t = \frac{P_t}{P_t^*}$ and PPP will hold. Hence

$$\begin{aligned}
\beta Cov_t \left[\frac{c_{t+1}}{c_t} \frac{P_t^*}{P_{t+1}^*}, (1+R_t)\frac{S_t}{S_{t+1}} \right] &= E_t \left[\beta \frac{c_{t+1}}{c_t} \frac{P_t^*}{P_{t+1}^*} (1+R_t) \frac{S_t}{S_{t+1}} \right] - E_t \left[\beta \frac{c_{t+1}}{c_t} \frac{P_t^*}{P_{t+1}^*} \right] E_t \left[(1+R_t) \frac{S_t}{S_{t+1}} \right] \\
&= (1+R_t) E_t \left[\beta \frac{c_{t+1}}{c_t} \frac{P_t}{P_{t+1}} \right] - (1+R_t) E_t \left[\beta \frac{c_{t+1}}{c_t} \frac{P_t}{P_{t+1}} \frac{S_{t+1}}{S_t} \right] E_t \left[\frac{S_t}{S_{t+1}} \right] \\
&= 1 - (1+R_t) \left\{ \begin{array}{c} E_t \left[\beta \frac{c_{t+1}}{c_t} \frac{P_t}{P_{t+1}} \right] E_t \left[\frac{S_{t+1}}{S_t} \right] \\ + Cov_t \left[\beta \frac{c_{t+1}}{c_t} \frac{P_t}{P_{t+1}}, \frac{S_{t+1}}{S_t} \right] \end{array} \right\} E_t \left[\frac{S_t}{S_{t+1}} \right] \\
&= 1 - \left\{ E_t \left[\frac{S_{t+1}}{S_t} \right] + \frac{1+R_t}{1+R_t^*} Cov_t \left[\beta \frac{c_{t+1}}{c_t} \frac{P_t}{P_{t+1}}, (1+R_t^*) \frac{S_{t+1}}{S_t} \right] \right\} E_t \left[\frac{S_t}{S_{t+1}} \right] \\
&= -E_t \left[\frac{1}{1+R_t^*} \frac{S_{t+1}}{S_t} \right] E_t \left[(1+R_t^*) \frac{S_{t+1}}{S_t} - (1+R_t) \right]
\end{aligned}$$

where we have assumed that $E_t \left[\frac{S_{t+1}}{S_t} \right] E_t \left[\frac{S_t}{S_{t+1}} \right] = 1$. As established by Siegel's inequality, this is not stricly correct as, to second-order aproximation, $E(\frac{1}{x}) \simeq \frac{1}{E(x)}[1+\frac{V(x)}{[E(x)]^2}] = \frac{1}{E(x)} \frac{E(x^2)}{[E(x)]^2} > \frac{1}{E(x)}$.

The foreign exchange risk premium for the foreign country is then

$$\begin{aligned}
E_t \left[(1+R_t) \frac{S_t}{S_{t+1}} - (1+R_t^*) \right] &= \beta (1+R_t^*) Cov_t \left[\frac{c_{t+1}}{c_t} \frac{P_t^*}{P_{t+1}^*}, (1+R_t) \frac{S_t}{S_{t+1}} \right] \\
&= -E_t \left[(1+R_t^*) \frac{S_{t+1}}{S_t} - (1+R_t) \right],
\end{aligned}$$

i.e. equal to the foreign exchange risk premium for the domestic country.

11.3. Let S_t denote the current price in dollars of one unit of foreign currency; $F_{t,T}$ is the delivery price agreed to in a forward contract; r is the domestic interest rate with continuous compounding; r^* is the foreign interest rate with continuous compounding.

(a) Consider the following pay-offs:

(i) investing in a domestic bond

(ii) investing a unit of domestic currency in a foreign bond and buying a forward contract to convert the proceeds.

Find the value of the forward exchange rate $F_{t,T}$.

(b) Suppose that the foreign interest rate exceeds the domestic interest rate at date t so that $r^* > r$. What is the relation between the forward and spot exchange rates?

Solution

(a) (i) Investing in a domestic bond gives the pay-off $e^{r(T-t)}$

(ii) A unit of domestic currency gives S_t^{-1} units of foreign currency and a pay-off in foreign currency of $S_t^{-1}e^{r^*(T-t)}$. The pay-off in terms of domestic currency is $\frac{F_{t,T}e^{r^*(T-t)}}{S_t}$. No-arbitrage implies that the pay-offs are the same, hence

$$e^{r(T-t)} = \frac{F_{t,T}e^{r^*(T-t)}}{S_t}$$

The value of the forward contract is therefore

$$F_{t,T} = S_t e^{(r-r^*)(T-t)}$$

(b) If $r^* > r$ then $F_{t,T} < S_t$.

11.4. (a) What is the price of a forward contract on a dividend-paying stock with stock price S_t?

(b) A one-year long forward contract on a non-dividend-paying stock is entered into when the stock price is \$40 and the risk-free interest rate is 10% per annum with continuous compounding. What is the forward price?

(c) Six months later, the price of the stock is \$45 and the risk-free interest rate is still 10%. What is the forward price?

Solution

(a) Assuming that dividends are paid *continuously* and the dividend yield is a constant q, at time T the pay-off to investing in S_t in a stock is $S_T e^{q(T-t)}$. Hence, a portfolio consisting of a long position in the forward contract involves paying $F_{t,T}$ for the stock and selling it for S_T. This yields a profit at time T of

$$\pi^1(T) = S_T - F_{t,T}.$$

A portfolio consisting of a long position in the stock and a short position in a risk-free asset involves borrowing $e^{-q(T-t)}S_t$. At the rate of interest r this costs $e^{(r-q)(T-t)}S_t$ at date T. Selling the stock for S_T then yields the profit

$$\pi^2(T) = S_T - e^{(r-q)(T-t)}S_t.$$

No-arbitrage implies that expected profits for the two portfolios are the same, hence

$$E_t(S_T) - F_{t,T} = E_t(S_T) - e^{(r-q)(T-t)}S_t.$$

The forward price for the stock is therefore

$$F_{t,T} = e^{(r-q)(T-t)}S_t.$$

(b) The forward price of a non-dividend paying stock is

$$F_{t,T} = e^{r(T-t)}S_t.$$

If $S_t = \$40$ and $r = 0.1$ then $F_{t,t+1} = \$e^{0.1}40 = \$44.2$.

(c) We now require $F_{t+\frac{1}{2},t+1} = \$e^{\frac{0.1}{2}\cdot\frac{1}{2}}40 = \$41.0$.

11.5. Suppose that in an economy with one and two zero-coupon period bonds investors maximize $E_t\Sigma_{s=0}^{\infty}\beta^s \ln c_{t+s}$. What is

(a) the risk premium in period t for the two-period bond, and

(b) its price in period t?

(c) What is the forward rate for the two-period bond?

(d) Hence, express the risk premium in terms of this forward rate.

Solution

(a) From the answers to Exercises 11.1 and 11.2 the Euler equations for one and two period bonds are

$$
\begin{aligned}
E_t\left[\beta\frac{c_{t+1}}{c_t}\frac{P_t}{P_{t+1}}\cdot(1+s_t)\right] &= 1 \\
E_t\left[\beta\frac{c_{t+1}}{c_t}\frac{P_t}{P_{t+1}}\cdot(1+h_{2,t+1})\right] &= 1
\end{aligned}
$$

where $1+s_t=\frac{1}{P_{1,t}}$, the holding-period return on the two-period bond is given by

$$
\begin{aligned}
1+h_{2,t+1} &= \frac{P_{1,t+1}}{P_{2,t}} \\
&= \frac{(1+R_{1,t})^{-(n-1)}}{(1+R_{2,t})^{-n}}
\end{aligned}
$$

$R_{n,t}$ is the yield to maturity on an n-period zero-coupon bond and $R_{1,t}=s_t$. Hence

$$
\begin{aligned}
0 &= E_t\left[\beta\frac{c_{t+1}}{c_t}\frac{P_t}{P_{t+1}}\cdot(h_{2,t+1}-s_t)\right] \\
&= E_t\left[\beta\frac{c_{t+1}}{c_t}\frac{P_t}{P_{t+1}}\right]E_t\left[h_{2,t+1}-s_t\right]+Cov_t\left[\beta\frac{c_{t+1}}{c_t}\frac{P_t}{P_{t+1}},h_{2,t+1}-s_t\right].
\end{aligned}
$$

It follows that the risk premium on the two-period bond is

$$
E_t\left[h_{2,t+1}-s_t\right]=-(1+s_t)Cov_t\left[\beta\frac{c_{t+1}}{c_t}\frac{P_t}{P_{t+1}},h_{2,t+1}-s_t\right].
$$

(b) The expected excess return can be written as

$$
E_t\left[1+h_{2,t+1}-(1+s_t)\right]=\frac{E_t[P_{1,t+1}]}{P_{2,t}}-(1+s_t).
$$

Hence the price of the two-period bond is

$$
P_{2,t}=\frac{E_t[P_{1,t+1}]}{(1+s_t)\left\{1-Cov_t\left[\beta\frac{c_{t+1}}{c_t}\frac{P_t}{P_{t+1}},h_{2,t+1}-s_t\right]\right\}},
$$

where $E_t[P_{1,t+1}]=E_t[\frac{1}{1+s_{t+1}}]$.

(c) The forward rate between periods $t+1$ and $t+2$ is obtained from

$$
1+f_{t,t+1}=\frac{P_{1,t}}{P_{2,t}}.
$$

(d) $P_{2,t}$ can be written in terms of the risk premium, which we denote by $\rho_{2,t}$, as

$$P_{2,t} = \frac{E_t[P_{1,t+1}]}{1 + s_t + \rho_{2,t}}.$$

It follows that

$$\begin{aligned} 1 + f_{t,t+1} &= \frac{P_{1,t}}{P_{2,t}} \\ &= \frac{P_{1,t}(1 + s_t + \rho_{2,t})}{E_t[P_{1,t+1}]}. \end{aligned}$$

Hence, the risk premium can be written as

$$\begin{aligned} \rho_{2,t} &= (1 + f_{t,t+1})\frac{E_t[P_{1,t+1}]}{P_{1,t}} - (1 + s_t) \\ &= (1 + f_{t,t+1})E_t(\frac{1 + s_t}{1 + s_{t+1}}) - (1 + s_t). \end{aligned}$$

11.6. Consider a Vasicek model with two independent latent factors z_{1t} and z_{2t}. The price of an n-period bond and the log discount factor may be written as

$$\begin{aligned} p_{n,t} &= -[A_n + B_{1n}z_{1t} + B_{2n}z_{2t}] \\ m_{t+1} &= -[z_{1t} + z_{2t} + \lambda_1 e_{1,t+1} + \lambda_2 e_{2,t+1}] \end{aligned}$$

where the factors are generated by

$$z_{i,t+1} - \mu_i = \phi_i(z_{it} - \mu_i) + e_{i,t+1}, \;\; i = 1, 2.$$

(a) Derive the no-arbitrage condition for an n-period bond and its risk premium. State any additional assumptions made.

(b) Explain how the yield on an n-period bond and its risk premium can be expressed in terms of the yields on one and two period bonds.

(c) Derive an expression for the n-period ahead forward rate.

(d) Comment on the implications of these results for the shape and behavior over time of the yield curve.

Solution

(a) The pricing equation for an n-period bond is

$$P_{nt} = E_t[M_{t+1}P_{n-1,t+1}].$$

Assuming that $P_{n,t}$ and M_{t+1} have a joint log-normal distribution with $p_{n,t} = \ln P_{n,t}$ and $m_{t+1} = \ln M_{t+1}$ gives

$$p_{nt} = E_t(m_{t+1} + p_{n-1,t+1}) + \frac{1}{2}V_t(m_{t+1} + p_{n-1,t+1}).$$

Hence, as $P_{0,t} = 1$, and so $p_{o,t} = 0$,

$$p_{1,t} = E_t m_{t+1} + \frac{1}{2}V_t(m_{t+1}).$$

The no-arbitrage condition is

$$E_t p_{n-1,t+1} - p_{n.t} + p_{1,t} + \frac{1}{2}V_t(p_{n-1,t+1}) = -Cov_t(m_{t+1}, p_{n-1,t+1})$$

and the risk premium is $-Cov_t(m_{t+1}, p_{n-1,t+1})$.

Evaluating

$$\begin{aligned} E_t[m_{t+1} + p_{n-1,t+1}] &= -[\Sigma_i z_{it} + A_{n-1} + \Sigma_i B_{i,n-1} E_t z_{i,t+1}] \\ &= -[\Sigma_i z_{it} + A_{n-1} + \Sigma_i B_{i,n-1}[\mu_i(1-\phi_i) + \phi_i z_{i,t}] \end{aligned}$$

where $\Sigma_i = \Sigma_{i=1}^2$ and

$$\begin{aligned} V_t[m_{t+1} + p_{n-1,t+1}] &= V_t[\Sigma_i(\lambda_i e_{i,t+1} + B_{i,n-1}e_{i,t+1})] \\ &= \Sigma_i(\lambda_i + B_{i,n-1})^2\sigma_i^2, \end{aligned}$$

the log pricing equation becomes

$$-[A_n + \Sigma_i B_{i,n} z_{i,t}] = -[A_{n-1} + \Sigma_i B_{i,n-1}\mu_i(1-\phi_i)] - \Sigma_i[1 + \phi_i B_{i,n-1} - \frac{1}{2}(\lambda_i + B_{i,n-1})^2\sigma_i^2)]z_{i,t}.$$

Equating terms on the left-hand and right-hand sides gives the recursive formulae

$$A_n = A_{n-1} + \Sigma_i B_{i,n-1}\mu_i(1-\phi_i)$$

$$B_{i,n} = 1 + \phi_i B_{i,n-1} - \frac{1}{2}(\lambda_i + B_{i,n-1})^2 \sigma_i^2$$

Using $p_{0,t} = 0$ we obtain $A_0 = 0$, $B_{i,0} = 0$, $B_{i,1} = 1 - \frac{1}{2}\lambda_i^2\sigma_i^2$ and $A_1 = 0$. Hence

$$\begin{aligned} s_t &= -p_{1,t} = A_1 + \Sigma_i B_{i,1} z_{i,t} \\ &= \Sigma_i (1 - \frac{1}{2}\lambda_i^2 \sigma_i^2) z_{i,t}. \end{aligned}$$

The no-arbitrage condition is therefore

$$E_t p_{n-1,t+1} - p_{nt} + p_{1,t} = \Sigma_i [-\frac{1}{2} B_{i,n-1}^2 \sigma_i^2 + \lambda_i B_{i,n-1} \sigma_i^2].$$

The first term is the Jensen effect and the second is the risk premium. Thus

$$\text{risk premium} = -Cov_t(m_{t+1}, p_{n-1,t+1}) = \Sigma_i \lambda_i B_{i,n-1} \sigma_i^2.$$

(b) The yield on an n-period bond is

$$R_{n,t} = -\frac{1}{n} p_{n,t} = \frac{1}{n}[A_n + \Sigma_i B_{i,n} z_{i,t}]$$

Hence

$$\begin{aligned} s_t &= -p_{1,t} = A_1 + \Sigma_i B_{i,1} z_{i,t} \\ R_{2,t} &= -\frac{1}{2} p_{2,t} = \frac{1}{2}[A_2 + \Sigma_i B_{i,2} z_{i,t}] \end{aligned}$$

are two equations in two unknowns $z_{1,t}$ and $z_{2,t}$, where A_n and $B_{i,n}$ are functions of the parameters. Thus, from the recursions

$$\begin{aligned} A_2 &= A_1 + \Sigma_i B_{i,1} \mu_i (1 - \phi_i) \\ &= \Sigma_i (1 - \frac{1}{2}\lambda_i^2 \sigma_i^2) \mu_i (1 - \phi_i) \end{aligned}$$

and

$$\begin{aligned} B_{i,2} &= 1 + \phi_i B_{i,1} - \frac{1}{2}(\lambda_i + B_{i,1})^2 \sigma_i^2 \\ &= 1 + \phi_i (1 - \frac{1}{2}\lambda_i^2 \sigma_i^2) - \frac{1}{2}(\lambda_i + 1 - \frac{1}{2}\lambda_i^2 \sigma_i^2)^2 \sigma_i^2. \end{aligned}$$

We can therefore solve for $z_{1,t}$ and $z_{2,t}$ as functions of the two yields s_t and $R_{2,t}$ - or any other yield. Knowing $z_{1,t}$ and $z_{2,t}$ we now can the solve for $R_{n,t}$ and the risk premium on an n-period bond.

(c) The n-period ahead forward rate is

$$\begin{aligned} f_{t,t+n} &= p_{n-1,t} - p_{n,t} \\ &= A_n - A_{n-1} + \Sigma_i (B_{i,n} - B_{i,n-1}) z_{i,t} \\ &= \Sigma_i B_{i,n-1} \mu_i (1-\phi_i) + \Sigma_i [-(1-\phi_i) B_{i,n-1} - \frac{1}{2}(\lambda_i + B_{i,n-1})^2 \sigma_i^2] z_{i,t}. \end{aligned}$$

This too can be expressed in terms of the yields s_t and $R_{2,t}$ - or any other yield.

(d) Broadly, the shape of a yield curve can be described in terms of its level and slope - or curvature. The level largely reflects the effects of inflation and the real interest rate both of which are captured in the short rate s_t. The slope largely reflects how inflation is expected to change in the future - up or down - and the greater risk at longer maturity horizons. Curvature reflects expected changes in the rate of inflation over time. We note that as, in effect, the first factor $z_{1,t}$ is s_t, the model captures this aspect of the yield curve well. The second factor $z_{2,t}$ has to capture all other features of the yield curve which, in general, it will be unable to do. In particular, it will be unable to pick up any curvature in the yield curve. Further, the risk premium varies with the time to maturity, but is independent of $z_{1,t}$ and $z_{2,t}$ and is fixed over time. This is a major weakness of the model.

11.7 In their affine model of the term structure Ang and Piazzesi (2003) specify the pricing kernel M_t directly as follows:

$$\begin{aligned}
M_{t+1} &= \exp(-s_t)\frac{\xi_{t+1}}{\xi_t} \\
s_t &= \delta_0 + \delta_1' z_t \\
z_t &= \mu + \phi' z_{t-1} + \Sigma e_{t+1} \\
\frac{\xi_{t+1}}{\xi_t} &= \exp(-\frac{1}{2}\lambda_t'\lambda_t - \lambda_t' e_{t+1}) \\
\lambda_t &= \lambda_0 + \lambda_1 z_t \\
p_{n,t} &= A_n + B_n' z_t
\end{aligned}$$

(a) Derive the yield curve, and

(b) and the risk premium on a n-period yield.

Solution

(a) We start with the asset pricing equation

$$P_{nt} = E_t[M_{t+1}P_{n-1,t+1}]$$

so that for $P_{n,t}$ and M_{t+1} jointly log-normal with $p_t = \ln P_t$ and $m_{t+1} = \ln M_{t+1}$ we have

$$p_{nt} = E_t(m_{t+1} + p_{n-1,t+1}) + \frac{1}{2}V_t(m_{t+1} + p_{n-1,t+1}).$$

As $p_{o,t} = 0$ and $s_t = -p_{1,t}$,

$$-s_t = E_t(m_{t+1}) + \frac{1}{2}V_t(m_{t+1})$$

Given the set-up of the exercise, it follows that $\delta_0 = -A_1$, $\delta_1 = -B_1$,

$$E_t(m_{t+1}) = -s_t - \frac{1}{2}V_t(m_{t+1})$$

and, since $E_t e_{t+1} e_{t+1}' = I$,

$$V_t(m_{t+1}) = \frac{1}{2}\lambda_t'\lambda_t.$$

Hence,

$$\begin{aligned} m_{t+1} &= -s_t + \Delta \ln \xi_{t+1} \\ &= -s_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t' e_{t+1} \\ &= -\delta_0 - \delta_1' z_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t' e_{t+1}. \end{aligned}$$

Evaluating p_{nt} we obtain

$$\begin{aligned} p_{nt} &= E_t(m_{t+1} + p_{n-1,t+1}) + \frac{1}{2}V_t(m_{t+1} + p_{n-1,t+1}) \\ A_n + B_n' z_t &= -\delta_0 - \delta_1' z_t - \frac{1}{2}\lambda_t'\lambda_t + A_{n-1} + B_{n-1}'(\mu + \phi' z_t) + \frac{1}{2}\lambda_t'\lambda_t + \frac{1}{2}B_{n-1}'\Sigma\Sigma' B_{n-1} + \lambda_t'\Sigma' B_{n-1} \\ &= -\delta_0 - \delta_1' z_t + A_{n-1} + B_{n-1}'(\mu + \phi' z_t) + \frac{1}{2}B_{n-1}'\Sigma\Sigma' B_{n-1} + B_{n-1}'\Sigma(\lambda_0 + \lambda_1 z_t) \end{aligned}$$

where

$$\begin{aligned} A_n &= -\delta_0 + A_{n-1} + B_{n-1}'\mu + \frac{1}{2}B_{n-1}'\Sigma\Sigma' B_{n-1} + B_{n-1}'\Sigma\lambda_0 \\ B_n &= -\delta_1 + \phi B_{n-1} + \frac{1}{2}B_{n-1}'\Sigma\Sigma' B_{n-1} + B_{n-1}'\Sigma\lambda_1. \end{aligned}$$

We note that, as required, $A_1 = -\delta_0$, $B_1 = -\delta_1$.

(b) The risk (term) premium is therefore

$$\begin{aligned} \text{risk premium} &= -Cov_t(m_{t+1}, p_{n-1,t+1}) \\ &= -B_{n-1}'\Sigma\lambda_t \\ &= -B_{n-1}'\Sigma(\lambda_0 + \lambda z_t) \end{aligned}$$

This depends on the factors z_t, which may be a mixture of observable and unobservable variables, and the coefficients B_{n-1} which depend on n, the time to maturity.

Chapter 12

12.1. According to rational expectations models of the nominal exchange rate, such as the Monetary Model, an increase in the domestic money supply is expected to cause an appreciation in the exchange rate, but the exchange rate depreciates. Explain why the Monetary Model is nonetheless correct.

Solution

The question is play on words. Nonetheless, it highlights an important feature of bi-lateral exchange rates, namely, that they are asset prices and hence respond instantaneously to new information. Both the spot rate and expected future exchange rates respond. As a result, the expected change $E_t \Delta s_{t+1}$ may, for example, increase but the spot rate may increase or decrease.

In the Monetary Model in Chapter 12 the exchange rate is determined from the uncovered interest parity condition

$$R_t = R_t^* + E_t \Delta s_{t+1},$$

which implies that

$$\begin{aligned} s_t &= E_t s_{t+1} + R_t^* - R_t \\ &= \sum\nolimits_{i=0}^{\infty} (R_{t+i}^* - R_{t+i}). \end{aligned}$$

From PPP and the two money markets, international differences in money supplies or output affect the interest differential and through this the (log) nominal exchange rate s_t. The solution for the exchange rate was shown to be

$$s_t = \left(\frac{\lambda}{1+\lambda}\right)^n E_t s_{t+n} + \frac{\lambda}{1+\lambda} \sum\nolimits_{i=0}^{n-1} \left(\frac{\lambda}{1+\lambda}\right)^i E_t \left[\tilde{m}_{t+i} - \tilde{y}_{t+i}\right]$$

where $\tilde{m}_t = m_t - m_t^*$, the log difference between the money supplies.

It follows that the effect on s_t of an unexpected increase in m_t - holding the other exogenous variables m_t^*, y_t and y_t^* fixed - may be obtained from

$$s_t = \frac{1}{1+\lambda} m_t > 0.$$

Hence the spot exchange rate jump depreciates. As the exchange rate returns to its original level next period,

$$E_t s_{t+1} - s_t = -\frac{1}{1+\lambda} m_t < 0$$

This implies that there is an expected appreciation of the exchange rate *next period* even though the spot rate is predicted by the Monetary Model to depreciate in the current period.

Put another way, the increase in m_t reduces R_t and raises s_t. It also decreases $R_t^* - R_t$ and hence, from the UIP condition, $E_t \Delta s_{t+1} = E_t s_{t+1} - s_t < 0$ - i.e. the exchange rate is expected to decrease.

12.2 The Buiter-Miller (1981) model of the exchange rate - not formally a DGE model but, apart from the backward-looking pricing equation, broadly consistent with such an interpretation - may be represented as follows:

$$\begin{aligned}
y_t &= \alpha(s_t + p_t^* - p_t) - \beta(R_t - E_t \Delta p_{t+1}) + g_t + \gamma y_t^* \\
m_t - p_t &= y_t - \lambda R_t \\
\Delta p_{t+1} &= \theta(y_t - y_t^n) + \pi_t^\# \\
E_t \Delta s_{t+1} &= R_t - R_t^*
\end{aligned}$$

where y is output, y^n is full employment output, g is government expenditure, s is the log exchange rate, R is the nominal interest rate, m is log nominal money, p is the log price level, $\pi^\#$ is target inflation and an asterisk denotes the foreign equivalent.

(a) Stating any assumptions you make, derive the long-run solution.

(b) Derive the short-run solution for the exchange rate.

(c) Hence comment on the effects of monetary and fiscal policy.

Solution

(a) Assuming that there is a steady-state solution, and this involves a constant rate of growth of the money supply equal to $\pi_t^\#$, and that g_t, y_t^*, y_t^n and R_t^* are constant in the long run, we can re-write the model in steady state as

$$
\begin{aligned}
y &= \alpha(s + p^* - p) - \beta(R - \pi^{\#}) + g + \gamma y^* \\
m - p &= y - \lambda R \\
\pi^{\#} &= \theta(y - y^n) + \pi^{\#} \\
0 &= R - R^*.
\end{aligned}
$$

It follows that $y = y^n$, $R = R^*$, $p = m - y^n + \lambda R^*$,

$$
\begin{aligned}
s &= p - p^* + \frac{1}{\alpha}(y^n - g - \gamma y^*) + \frac{\beta}{\alpha}(R^* - \pi^{\#}) \\
&= m - \frac{\beta}{\alpha}\pi^{\#} - \frac{1-\alpha}{\alpha}y^n - \frac{1}{\alpha}(g + \gamma y^*) - p^* + (\lambda + \frac{\beta}{\alpha})R^*.
\end{aligned}
$$

(b) To obtain the short-run solution first we reduce the full model to two equations - in p_t and s_t - by eliminating the other endogenous variables. First, eliminate R_t using the UIP condition. Second, eliminate y_t from the money demand and price equations. This gives the first equation below. Third, eliminate y_t from the money demand and demand equations. This gives the second equation.

$$
\begin{aligned}
p_{t+1} - (1 - \theta\lambda)p_t - \theta\lambda E_t s_{t+1} + \theta\lambda s_t &= a_t \\
\beta E_t p_{t+1} - (\beta + \lambda)E_t s_{t+1} + (1 - \alpha - \beta)p_t + (\alpha + \beta + \lambda)s_t &= b_t
\end{aligned}
$$

where

$$
\begin{aligned}
a_t &= \theta(m_t + \lambda R_t^* - y_t^n) + \pi_t^{\#} - \varepsilon_{t+1} \\
b_t &= m_t - g_t - \gamma y_t^* - \alpha p_t^* + (\beta + \lambda)R_t^*
\end{aligned}
$$

and $\varepsilon_{t+1} = p_{t+1} - E_t p_{t+1}$.

Introducing the lag operator $Lx_t = x_{t-1}$ and $L^{-1}x_t = E_t x_{t+1}$, we can write the model in matrix notation as

$$
L^{-1}\begin{bmatrix} 1 - (1 - \theta\lambda)L & -\theta\lambda + \theta\lambda L \\ \beta + (1 - \alpha - \beta)L & -(\beta + \lambda) + (\alpha + \beta + \lambda)L \end{bmatrix}\begin{bmatrix} p_t \\ s_t \end{bmatrix} = \begin{bmatrix} a_t \\ b_t \end{bmatrix}
$$

or as

$$L^{-1}\left\{\begin{bmatrix} 1 & -\theta\lambda \\ \beta & -\beta-\lambda \end{bmatrix} - \begin{bmatrix} 1-\theta\lambda & -\theta\lambda \\ \alpha+\beta-1 & -(\alpha+\beta+\lambda) \end{bmatrix} L\right\}\begin{bmatrix} p_t \\ s_t \end{bmatrix} = \begin{bmatrix} a_t \\ b_t \end{bmatrix}$$

We now use the result in the Appendix that if

$$B(L)Z_t = z_{t-1}$$

where $B(L) = I - AL$ then the eigenvalues can be obtained from

$$\det B(L) = 0$$

We also note that

$$\begin{aligned} \det(I - AL) &= (1-\lambda_1 L)(1-\lambda_2 L) \\ &= 1 - (\lambda_1 + \lambda_2)L + \lambda_1\lambda_2 L^2 \\ &= 1 - (trA)L + (\det A)L^2 \end{aligned}$$

where the roots $\gamma_i = \frac{1}{\lambda_i}$ are obtained from

$$\{\lambda_1,\ \lambda_2\} = \frac{1}{2}trA \pm \frac{1}{2}[(trA)^2 - 4(\det A)]^{\frac{1}{2}}$$

and are approximately

$$\{\lambda_1,\ \lambda_2\} \simeq \{\frac{\det A}{trA},\ \ trA - \frac{\det A}{trA}\}.$$

We also note that if $\det(I - AL)|_{L=1} < 0$, then the solution is a saddlepath.

The model can be written as

$$(P - QL)Z_{t+1} = z_t,$$

or as

$$(I - P^{-1}QL)Z_{t+1} = P^{-1}z_t.$$

We require $\det(I - AL)|_{L=1} \equiv \det(I - P^{-1}QL)_{L=1}$. We note that

$$\det(I - P^{-1}Q) = (\det P)^{-1}\det(P - Q)$$

Hence

$$\begin{aligned}
\det(I - AL)|_{L=1} &\equiv \left(\det \begin{bmatrix} 1 & -\theta\lambda \\ \beta & -\beta - \lambda \end{bmatrix}\right)^{-1} \det \begin{bmatrix} \theta\lambda & 0 \\ 1-\alpha & \alpha \end{bmatrix} \\
&= \frac{\alpha\theta\lambda}{\beta\theta\lambda - \beta - \lambda}
\end{aligned}$$

The model would therefore have a saddlepath solution if $\det P = \beta\theta\lambda - \beta - \lambda < 0$ - i.e. if $\beta^{-1} + \lambda^{-1} > \theta$. It can shown that this is NOT the case as assuming a saddlepath solution gives the wrong signs on the effects of monetary and fiscal policy. Instead we assume that $\lambda_1, \lambda_2 > 1$

Noting that the inverse of $B(L)$ is the adjoint matrix of $B(L)$ divided by the determinant of $B(L)$

$$B(L)^{-1} = \frac{adjB(L)}{\det B(L)}$$

we can write the model as

$$\begin{aligned}
\det(I - P^{-1}QL)Z_{t+1} &= (1 - \lambda_1 L)(1 - \lambda_2 L)Z_{t+1} \\
&= \frac{adj(P - QL)}{\det P} z_t
\end{aligned}$$

where

$$adj(P - QL) = \begin{bmatrix} -(\beta + \lambda) + (\alpha + \beta + \lambda)L & \theta\lambda - \theta\lambda L \\ -\beta - (1 - \alpha - \beta)L & 1 - (1 - \theta\lambda)L \end{bmatrix}$$

and $\det P = \beta\theta\lambda - \beta - \lambda$. Assuming that λ_1,$\lambda_2 > 1$ we therefore have

$$\lambda_1\lambda_2 L^2(1 - \lambda_1^{-1}L^{-1})(1 - \lambda_2^{-1}L^{-1})Z_{t+1} = \frac{adj(P - QL)}{\det P} z_t$$

Thus

$$\begin{aligned}
Z_{t+1} &= \frac{1}{\lambda_1 + \lambda_2} L^{-2} [\frac{\lambda_1^{-1}}{1 - \lambda_1^{-1}L^{-1}} + \frac{\lambda_2^{-1}}{1 - \lambda_2^{-1}L^{-1}}] \frac{adj(P - QL)}{\det P} z_t \\
&= \frac{1}{\lambda_1 + \lambda_2} \Sigma_{i=0} [\lambda_1^{-(i+1)} L^{-(i+2)} + \lambda_2^{-(i+1)} L^{-(i+2)}] \frac{adj(P - QL)}{\det P} z_t
\end{aligned}$$

It can now be shown that the solution for the exchange rate in response to the exogenous variables

m_t and g_t is

$$
\begin{aligned}
s_t &= \frac{1}{(\lambda_1+\lambda_2)(\beta+\lambda-\beta\theta\lambda)}\Sigma_{i=0}[\lambda_1^{-(i+1)}L^{-(i+1)}+\lambda_2^{-(i+1)}L^{-(i+1)}]\{[(1-\theta\beta)+L]m_t \\
&\quad +[(1-(1-\theta\lambda)L]g_t\} \\
&= \frac{1}{(\lambda_1+\lambda_2)(\beta+\lambda-\beta\theta\lambda)}\left\{\begin{array}{c} \Sigma_{i=0}\{[\lambda_1^{-(i+1)}+\lambda_2^{-(i+1)}][[1-\theta\lambda-\theta(1-\alpha-\beta)]E_t m_{t+i} \\ +(1-\theta\beta)E_t m_{t+i+1}] \\ -\Sigma_{i=0}\{[\lambda_1^{-(i+1)}+\lambda_2^{-(i+1)}][(1-\theta\lambda)E_t g_{t+i}-E_t g_{t+i+1}] \end{array}\right\}
\end{aligned}
$$

(c) It can now be seen that in the long run an increase in the money supply causes the exchange rate to depreciate due to a fall in the doemstic interest rate, and an increase in government expenditures causes the exchange rate to appreciate due to a rise in the interest rate. The impact effect of an increase in m_t is to depreciate the exchange rate if $1-\theta\lambda-\theta(1-\alpha-\beta)>0$. The impact effect of an increase in g_t is to appreciate the exchange rate if $1-\theta\lambda>0$. But an anticipated increase in g_{t+1} would cause the exchange rate to depreciate in period t in order that it would be expected to appreciate between periods t and $t+1$ as a result of the change in g_{t+1}.

12.3 Consider a small cash-in-advance open economy with a flexible exchange rate in which output is exogenous, there is Calvo pricing, PPP holds in the long run, UIP holds and households maximize $\Sigma_{j=0}^{\infty}\beta^j\ln c_{t+j}$ subject to their budget constraint

$$S_t\Delta F_{t+1}+\Delta M_{t+1}+P_t c_t=P_t x_t+R_t^* S_t F_t$$

where P_t is the general price level, c_t is consumption, x_t is output, F_t is the net foreign asset position, M_t is the nominal money stock, S_t is the nominal exchange rate and R_t^* is the foreign nominal interest rate.

(a) Derive the steady-state solution of the model when output is fixed.

(b) Obtain a log-linear approximation to the model suitable for analysing its short-run behavior

(c) Comment on its dynamic properties.

(a) The problem for the domestic economy is to maximize $\Sigma_{j=0}^{\infty}\beta^j \ln c_{t+j}$ with respect to consumption, money and net foreign asset holdings subject to the budget constraint and the cash-in-advance constraint. The Lagrangian is

$$\mathcal{L} = \sum_{j=0}^{\infty} \left\{ \begin{array}{c} \beta^j \ln c_{t+j} + \lambda_{t+j}[P_{t+j}x_{t+j} + (1 + R^*_{t+j})S_{t+j}F_{t+j} + M_{t+j} \\ -S_{t+j}F_{t+j+1} - M_{t+j+1} - P_{t+j}c_{t+j}] + \mu_{t+j}(M_{t+j} - P_{t+j}c_{t+j}) \end{array} \right\}.$$

The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_{t+j}} &= \frac{\beta^j}{c_{t+j}} - (\lambda_{t+j} + \mu_{t+j})P_{t+j} = 0 \qquad j \geq 0 \\
\frac{\partial \mathcal{L}}{\partial F_{t+j}} &= \lambda_{t+j}(1 + R^*_{t+j})S_{t+j} - \lambda_{t+j-1}S_{t+j-1} = 0 \qquad j > 0 \\
\frac{\partial \mathcal{L}}{\partial M_{t+j}} &= \lambda_{t+j} - \lambda_{t+j-1} + \mu_{t+j} = 0 \qquad j > 0.
\end{aligned}$$

The consumption Euler equation is therefore

$$\frac{\beta c_t P_t S_{t+1}(1 + R^*_t)}{P_{t+1}S_t c_{t+1}} = \frac{\beta c_t P_t(1 + R_t)}{P_{t+1}c_{t+1}} = \frac{\beta c_t(1 + r_t)}{c_{t+1}} = 1.$$

where r_t is the real interest rate defined by

$$1 + r_t = \frac{1 + R_t}{1 + \frac{\Delta P_{t+1}}{P_t}}.$$

In steady state, $\Delta c_{t+1} = 0$ and , if $\beta = \frac{1}{1+\theta}$, then $r_t = \theta$. The steady-state level of consumption is

$$c = x + RSf$$

where $x_t = x$ and $f = \frac{F}{P}$ is the real foreign asset position.

(b) In the short run the economy is described by the following equations

$$\begin{aligned}
\frac{\beta c_t P_t(1 + R_t)}{P_{t+1}c_{t+1}} &= 1 \\
S_t\Delta F_{t+1} + \Delta M_{t+1} + P_t c_t &= P_t x_t + R^*_t S_t F_t \\
M_t &= P_t c_t \\
\pi_t &= \frac{\rho(1-\gamma)}{1-\rho\gamma}(s_t + p^*_t - p_{t-1}) + \frac{\gamma}{1-\rho\gamma}\pi_{t+1} \\
1 + R_t &= \frac{S_{t+1}(1 + R^*_t)}{S_t},
\end{aligned}$$

where the fourth equation is the Calvo pricing equation (see Ch 9 for the definition of the parameters) and the fifth equation is UIP; $p_t = \ln P_t$, $\pi_t = \Delta p_t$ and $s_t + p_t^*$ is the "desired" value of p_t, which is the logarithm of the PPP price where $s_t = \ln S_t$ and $p_t^* = \ln P_t^*$, the world price. In the pricing equation and UIP the conditional expectation has been omitted.

A log-linear approximation to the model about $\pi = R^* = 0$ is

$$\begin{aligned}
\Delta \ln c_{t+1} &= R_t + \pi_{t+1} - \theta \\
\frac{Sf}{c}\Delta \ln f_{t+1} + \frac{m}{c}\Delta \ln m_{t+1} + \ln c_t + \frac{Sf+m}{c}\pi_{t+1} &= \frac{x}{c}\ln x_t + \frac{Sf}{c}R_t \\
\pi_t &= \frac{\rho(1-\gamma)}{1-\rho\gamma}(s_t + p_t^* - p_{t-1}) + \frac{\gamma}{1-\rho\gamma}\pi_{t+1} \\
R_t &= R_t^* + \Delta s_{t+1}
\end{aligned}$$

where $m_t = \frac{M_t}{P_t}$ and $\frac{m}{c} = 1$.

The dynamic behavior of the model must be analysed numerically due to the number of roots the model has. The presence of both leading and lagged terms indicates that the solution to the model will have both forward and backward looking components and, almost certainly, a saddlepath solution. The backward-looking component is due to the Calvo pricing equation. If $\rho = 0$ then the solution would be just forward looking. This is when the probability of being able to adjust prices in any period is zero.

Numerical analysis shows, for example, that a temporary increase in domestic money causes a temporary depreciation of the exchange rate, a temporary increase in the domestic price level and consumption and a temporary fall in the domestic interest rate. A temporary increase in foreign prices (i.e. to the target price level) causes a temporary appreciation of the exchange rate, a temporary increase in the domestic price level and interest rate and a temporary fall in consumption. In each case the movement in domestic prices is small and the exchange rate is large showing the key role of a floating exchange rate in smoothing domestic prices from such shocks.

12.4. Suppose the global economy consists of two identical countries who take output as given, have cash-in-advance demands for money based on the consumption of domestic and foreign goods and services, and who may borrow or save either through domestic or foreign bonds. Purchasing power parity holds and the domestic and foreign money supplies are exogenous. Global nominal bond holding satisfies $B_t + S_t B_t^* = 0$ where S_t is the domestic price of nominal exchange and B_t is the nominal supply of domestic bonds. The two countries maximize $\Sigma_{j=0}^{\infty}\beta^j \ln c_{t+j}$ and $\Sigma_{j=0}^{\infty}\beta^j \ln c_{t+j}^*$, respectively, where c_t is real consumption. Foreign equivalents are denoted with an asterisk.

(a) Derive the solutions for consumption and the nominal exchange rate.

(b) What are the effects of increases in

(i) the domestic money supply and

(ii) domestic output?

Solution

(a) Let the two countries be a (the domestic economy) and b (the foreign economy). Consider first the domestic economy. The nominal budget constraint is

$$\Delta B_{t+1}^a + S_t \Delta B_{t+1}^{a*} + \Delta M_{t+1}^a + S_t \Delta M_{t+1}^{a*} + P_t c_t^a + S_t P_t^* c_t^{a*} = P_t x_t + R_t B_t^a + S_t R_t^* B_t^{a*}$$

where P_t is the general price level, c_t^a and c_t^{a*} are domestic consumption of domestic and foreign goods and services, B_t^a and B_t^{a*} are domestic nominal bond holdings, M_t^a and M_t^{a*} are domestic nominal money holdings required for purchasing domestic and foreign goods and services, respectively, and R_t and R_t^* are the domestic and foreign nominal interest rates.

PPP - or the law of one price as there is only one good produced in each country - implies that $P_t = S_t P_t^*$ and the cash-in-advance constraints give $P_t c_t^a = M_t^a$ and $P_t^* c_t^{a*} = M_t^{a*}$. Further, $c_t = c_t^a + c_t^{a*}$.

The problem for the domestic economy therefore is to maximize with respect to consumption and bond holdings the Lagrangian

$$\mathcal{L}=\sum_{j=0}^{\infty}\left\{\begin{array}{c}\beta^j \ln c_{t+j}+\lambda_{t+j}[P_{t+j}x_{t+j}+(1+R_{t+j})B^a_{t+j}+S_{t+j}(1+R^*_{t+j})B^{a*}_{t+j}+M^a_{t+j}+S_{t+j}M^{a*}_{t+j}\\ -B^a_{t+j+1}-S_{t+j}B^{a*}_{t+j+1}-M^a_{t+j+1}-S_{t+j}\Delta M^{a*}_{t+j+1}-P_{t+j}c^a_{t+j}-S_{t+j}P^*_{t+j}c^{a*}_{t+j}]\\ +\mu_{t+j}(M^a_{t+j}-P_{t+j}c^a_{t+j})+\nu_{t+j}(S_{t+j}M^{a*}_{t+j}-S_{t+j}P^*_{t+j}c^{a*}_{t+j})\end{array}\right\}.$$

The first-order conditions are

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial c^a_{t+j}} &= \frac{\beta^j}{c_{t+j}}-(\lambda_{t+j}+\mu_{t+j})P_{t+j}=0 \qquad j\geq 0\\ \frac{\partial\mathcal{L}}{\partial c^b_{t+j}} &= \frac{\beta^j}{c_{t+j}}-(\lambda_{t+j}+\nu_{t+j})S_{t+j}P^*_{t+j}=0 \qquad j\geq 0\\ \frac{\partial\mathcal{L}}{\partial B^a_{t+j}} &= \lambda_{t+j}(1+R_{t+j})-\lambda_{t+j-1}=0 \qquad j>0\\ \frac{\partial\mathcal{L}}{\partial B^{a*}_{t+j}} &= \lambda_{t+j}S_{t+j}(1+R^*_{t+j})-\lambda_{t+j-1}S_{t+j-1}=0 \qquad j>0\\ \frac{\partial\mathcal{L}}{\partial M^a_{t+j}} &= \lambda_{t+j}-\lambda_{t+j-1}+\mu_{t+j}=0 \qquad j>0\\ \frac{\partial\mathcal{L}}{\partial M^{a*}_{t+j}} &= \lambda_{t+j}-\lambda_{t+j-1}+\nu_{t+j}S_{t+j}=0 \qquad j>0.\end{aligned}$$

It follows from the third and fourth equations that

$$1+R_{t+1}=\frac{S_{t+1}(1+R^*_{t+1})}{S_t}$$

which is the uncovered interest parity condition (based on perfect foresight). Given PPP, the consumption Euler equation is

$$\frac{\beta c_t P_t(1+R_t)}{P_{t+1}c_{t+1}}=\frac{\beta c_t(1+r_t)}{c_{t+1}}=1.$$

where r_t is the real interest rate defined by

$$1+r_t=\frac{1+R_t}{1+\frac{\Delta P_{t+1}}{P_t}}.$$

For the foreign country the budget constraint is

$$\Delta B^{b*}_{t+1}+\Delta B^b_{t+1}/S_t+\Delta M^{b*}_{t+1}+\Delta M^b_{t+1}/S_t+P^*_tc^{b*}_t+P_tc^b_t/S_t=P^*_tx^*_t+R^*_tB^{b*}_t+R_tB^b_t/S_t,$$

and the Euler equation is

$$\frac{\beta c^*_t(1+r^*_t)}{c^*_{t+1}}=1.$$

In steady state, $\Delta c_{t+1} = \Delta c^*_{t+1} = 0$ and , if $\beta = \frac{1}{1+\theta}$, then $r_t = r^*_t = \theta$. It follows from UIP that

$$\begin{aligned}
\frac{P_t}{P_{t+1}}(1 + R_{t+1}) &= \frac{P^*_t}{P^*_{t+1}}(1 + R^*_{t+1}) \\
&= \frac{S_t P^*_t}{S_{t+1} P^*_{t+1}}(1 + R_{t+1})
\end{aligned}$$

or

$$\frac{P_t}{P_{t+1}} = \frac{S_t P^*_t}{S_{t+1} P^*_{t+1}}$$

which is relative PPP; it also follows from PPP.

Noting that $B_t = B^a_t + B^b_t$, $B^*_t = B^{a*}_t + B^{b*}_t$, $M_t = M^a_t + M^b_t$ and $M^*_t = M^{a*}_t + M^{b*}_t$, the sum of the two budget constraints is

$$B_{t+1} + S_t B^*_{t+1} + \Delta M_{t+1} + S_t \Delta M^*_{t+1} + P_t c_t + S_t P^*_t c^*_t = P_t x_t + S_t P^*_t x^*_t + (1 + R_t) B_t + S_t (1 + R^*_t) B^*_t.$$

Hence, from the condition $B_t + S_t B^*_t = 0$ and PPP,

$$c_t + c^*_t = x_t + x^*_t - \Delta m_{t+1} - \pi_{t+1} m_{t+1} - \Delta m^*_{t+1} - \pi^*_{t+1} m^*_{t+1}.$$

where $m_t = \frac{M_t}{P_t}$, $\pi_{t+1} = \frac{\Delta P_{t+1}}{P_t}$ and an asterisk denotes the foreign equivalent. Thus, total "world" consumption equals total "world" income less changes in domestic and real money balances and the effects of domestic and foreign inflation in eroding real money balances. In steady state, the changes in real balances are zero and so world consumption equals total world income less inflation effects on real balances.

The nominal exchange rate is obtained from the relative money supplies of the two countries (which equals their relative money demands) which, as PPP holds and consumption is the same in each country, is

$$\begin{aligned}
\frac{M_t}{M^*_t} &= \frac{M^a_t + M^b_t}{M^{a*}_t + M^{b*}_t} \\
&= \frac{P_t c^a_t + P_t c^{b*}_t}{P^*_t c^{a*}_t + P^*_t c^b_t} = \frac{P_t (c^a_t + c^{b*}_t)}{P^*_t (c^{a*}_t + c^b_t)} = S_t \frac{c^a_t + c^{b*}_t}{c^{a*}_t + c^b_t}
\end{aligned}$$

Hence

$$S_t = \frac{M_t}{M_t^*} \frac{c_t^{a*} + c_t^b}{c_t^a + c_t^{b*}}.$$

If the two countries had identical incomes then this would reduce to $S_t = \frac{M_t}{M_t^*}$.

(b) (i) An increase in the domestic money supply would raise S_t proportionally, i.e. cause a depreciation of the exchange rate. It would also cause a corresponding increase in the domestic price level, thereby maintaining the real domestic money supply. Consumption would remain unchanged.

(ii) As utility is a function of total output, domestic and foreign output are perfect substitutes. Consequently, the domestic economy can choose to consume only domestic output and not import from abroad. An increase in domestic income would cause a corresponding increase in domestic consumption. This would raise the demand for real money. As the nominal money supply is unchanged, a fall in the domestic price level is required. Since PPP holds, the nominal exchange rate S_t must therefore fall, i.e. the exchange rate appreciates.

12.5 Consider a world consisting of two economies A and B. Each produces a single tradeable good and issues a risky one-period bond with real rate of returns r_t^A and r_t^B, respectively.

(a) Express the real exchange rate e_t between these countries in terms of their marginal utilities.

(b) Derive the real interest parity condition.

(c) How is this affected in the following cases:

(i) both countries are risk neutral,

(ii) markets are complete?

Solution

(a) Each country maximizes inter-temporal utility subject to their nominal budget constraint then each satisfies the first-order conditions

$$\begin{aligned} \beta^j U'_{t+j} &= \lambda_{t+j} P_{t+j} \qquad j \geq 0 \\ \lambda_{t+j}(1+R_{t+j}) - \lambda_{t+j-1} &= 0 \qquad j > 0 \end{aligned}$$

where U' is marginal utility, P is the price level, R is the nominal return and λ is the Lagrange multiplier (the marginal utility of wealth). Hence

$$\frac{P_t^A}{P_t^B} = \frac{U_t'^A}{U_t'^B} \frac{\lambda_t^B}{\lambda_t^A}$$

If S_t is the nominal exchange rate (the price of country A's currency in terms of country B's) then the real exchange rate is

$$e_t = \frac{S_t P_t^A}{P_t^B} = S_t \frac{U_t'^A}{U_t'^B} \frac{\lambda_t^B}{\lambda_t^A}$$

(b) Eliminating the Lagrange multipliers from the two first-order conditions gives the Euler equation

$$\frac{\beta U'_{t+1} P_t (1+R_{t+1})}{U'_t P_{t+1}} = \frac{\beta U'_{t+1}(1+r_{t+1})}{U'_t} = 1$$

where

$$1 + r_{t+1} = \frac{P_t(1+R_{t+1})}{P_{t+1}}$$

hence r^A and r^B are related through

$$\frac{\beta^A U_{t+1}'^A (1+r_{t+1}^A)}{U_t'^A} = \frac{\beta^B U_{t+1}'^B (1+r_{t+1}^B)}{U_t'^B}.$$

or

$$\frac{1+r_{t+1}^A}{1+r_{t+1}^B} = \frac{\beta^B U_{t+1}'^B / U_t'^B}{\beta^A U_{t+1}'^A / U_t'^A}.$$

(c) (i) Under risk neutrality $U_t^{'} = c_t$. Hence

$$\frac{1+r_{t+1}^A}{1+r_{t+1}^B} = \frac{\beta^B c_{t+1}^B / c_t^B}{\beta^A c_{t+1}^A / c_t^A}$$

(ii) In complete markets $U_t^{'A} = U_t^{'B}$ and $\beta^A = \beta^B$. Hence

$$r_{t+1}^A = r_{t+1}^B.$$

Chapter 13

13.1. Consider the following characterizations of the IS-LM and DGE models:

IS-LM

$$\begin{aligned} y &= c(y,r) + i(y,r) + g \\ m - p &= L(y,r) \end{aligned}$$

DGE

$$\begin{aligned} \Delta c &= -\frac{1}{\sigma}(r - \theta) = 0 \\ y &= c + i + g \\ y &= f(k) \\ \Delta k &= i \\ f_k &= r \end{aligned}$$

where y is output, c is consumption, i is investment, k is the capital stock, g is government expenditure, r is the real interest rate, m is log nominal money and p is the log price level.

(a) Comment on the main differences in the two models and on the underlying approaches to macroeconomics.

(b) Comment on the implications of the two models for the effectiveness of monetary and fiscal policy.

Solution

(a) The first two equations are IS and LM equations denoting equilibrium in the goods and money markets respectively. They incorporate the consumption and investment functions and the demand for money. The IS and LM equations determine y and r. It is implicitly assumed that there is no inflation, hence the real interest rate is present instead of the nominal interest rate.

The principal additional feature of the DGE model is the inclusion of capital. This is a crucial difference. Whereas equilibrium in the DGE model is unique, involving a single value of the capital stock, equilibrium in the IS-LM model is not unique as it is consistent with an infinity of values of the capital stock. The equilibrium in the DGE model is known as a stock equilibrium, and that in the IS-LM model is a flow equilibrium.

The determination of the capital stock involves an inter-temporal decision, i.e. concern not just for the present but also the future. There would no point in retaining any capital after the current period if there were no future. The presence of investment in the IS-LM model and the absence of capital indicates an emphasis on the present unaffected by future considerations. Consequently, the IS-LM model is at best suited to the short-term, whereas the DGE model is also appropriate for the longer term and, in particular, for the analysis of growth.

These distinctions arise from a more fundamental difference in approach to macroeconomics. The key issue dealt with in the DGE model is that of consumption today or consumption in the future. The role of investment, and hence capital, is that of transferring resources from consumption today to consumption in the future. The real interest rate plays a key role in this decision as it determines whether or not deferring consumption increases utility. The real interest rate is obtained from the productivity of capital rather than monetary policy.

(b) The Keynesian IS-LM model was developed in an era of low inflation. An increase in the nominal money supply was therefore, in effect, an increase in the real money supply and, according to the IS-LM model, this has real effects - on output and the real interest rate. In particular, a monetary expansion raises output.

Money is absent in the RBC model. When introduced in the form of a cash-in-advance constraint, changes in nominal money have no effect on real variables due to a corresponding change in the price level that leaves real money balances unchanged. Thus monetary policy is ineffective.

Accumulated empirical evidence showed that in fact money did have real effects, but only in the short term - for about 18 months - until the price level caught up, after which the predictions of the RBC model held. The second generation of DGE models (or dynamic stochastic general equilibrium - DSGE - models, if allowance is made for uncertainty), which introduced sticky prices, were largely designed to deal with this problem.

Fiscal policy in the IS-LM model has a strong real effect. This is because both the household and the government budget constraints are ignored. It is implicitly assumed that any government deficit is either bond or money financed but the need to redeem bonds in the future, and the inflation tax associated with nominal money expansion, are also ignored.

In the DGE model above, an increase in government expenditures crowd out consumption one for one, and there is no effect on output. This is because, implicitly, households perceive that taxes will rise in the future to pay for the increase in government expenditures. Fiscal policy is therefore ineffective in this DGE model.

Evidence on whether in fact fiscal policy has any effect on output is still unclear and divides professional opinion. The case for its effectiveness requires a weakening of some of the assumptions of this DGE model. For example, perhaps some households are myopic about the future, or some households face borrowing constraints and are therefore unable to smooth consumption, or markets fail to clear resulting in unemployed resources, especially of labor. Once again, these are essentially short-term phenomena.

13.2. (a) How might a country's international monetary arrangements affect its conduct of monetary policy?

(b) What other factors might influence the way it carries out its monetary policy?

Solution

(a) The prime aims of international monetary arrangements are to facilitate economic activity and maintain competitiveness. Some types of arrangements also control prices and inflation and supplant domestic monetary policy. Others require a separate monetary policy.

The main types of internationbal monetary arrangements are the gold standard, and fixed or floating exchange rates. Under the gold standard, competitiveness is automatically maintained by fluctuations in the price level. If prices are not sufficiently flexible, recessions may occur whilst prices adjust. There is no tendency for inflation unless the gold supply increases either as result of domestic or world gold extraction, or domestic accumulation of gold through persistent trade surpluses - or, historically, as bounty. The role of monetary policy under the gold standard is to maintain a fixed parity between gold and fiat currency.

Under fixed exchange rates domestic currency has a fixed rate of exchange rate with other currencies. In practice, domestic currency is tied to a world reserve currency, currently the US dollar. This implies that US monetary policy is not necessarily tied to anything and is at the discretion of the US. The monetary policy of countries tied to the dollar is curtailed; it is simply to maintain the fixed parity. This requires maintaining competitiveness using interest rates. The euro system is a slightly different fixed exchange rate arrangement, involving a common currency administered by a federal body, the ECB. As in the US federal system, the same currency is used in each country (state).

Under a floating rate system a country has no foreign nominal anchor and must use its monetary policy to provide one. In effect, each country therefore becomes like the US in this respect.

(b) If, in the long run, inflation is entirely a monetary phenomenon, long-run monetary policy must be to control inflation. In the short run, however, as monetary policy has real effects, it may also be used for stabilization policy. This can be with the aim of controlling domestic demand or the exchange rate, and hence competitiveness and trade. The remits of central banks differ in the extent to which monetary policy is allowed to be used for stabilization policy. For example, the Bank of England and the ECB act as strict inflation targeters while the US Fed acts as a flexible inflation targeter.

13.3. The Lucas-Sargent proposition is that systematic monetary policy is ineffective. Examine this hypothesis using the following model of the economy due to Bull and Frydman (1983):

$$\begin{aligned} y_t &= \alpha_1 + \alpha_2(p_t - E_{t-1}p_t) + u_t \\ d_t &= \beta(m_t - p_t) + v_t \\ \Delta p_t &= \theta(p_t^* - p_{t-1}), \end{aligned}$$

where y is output, d is aggregate demand, p is the log price level, p^* is the market clearing price, m is log nominal money and u and v are mean zero mutually and serially independent shocks.

(a) Derive the solutions for output and prices.

(b) If $m_t = \mu + \varepsilon_t$ where ε_t is a mean zero, serially independent shock, comment on the effect on prices of

(i) an unanticipated shock to money in period t,

(ii) a temporary anticipated shock to money in period t,

(iii) a permanent anticipated shock to money in period t.

(c) Hence comment on the Lucas-Sargent proposition.

Solution

(a) Under market clearing $y_t = d_t$ hence the market-clearing price evolves according to

$$(\alpha_2 + \beta)p_t^* - \alpha_2 E_{t-1}p_t^* = -\alpha_1 + \beta m_t - u_t + v_t = -\alpha_1 + e_t.$$

If the solution to p_t^* is $p_t^* = \gamma + A(L)e_t$ then

$$\beta\gamma + (\alpha_2 + \beta)A(L) - \alpha_2[A(L) - a_0)]e_t = -\alpha_1 + e_t.$$

Hence,

$$\begin{aligned} A(L) &= \frac{1 - \alpha_2 a_0}{\beta} \\ \gamma &= -\frac{\alpha_1}{\beta}. \end{aligned}$$

The solution is therefore

$$p_t^* = -\frac{\alpha_1}{\beta} + \frac{1 - \alpha_2 a_0}{\beta} e_t.$$

Due to the presence of a_0 the solution is not fully determined.

To derive the solution for p_t we substitute the solution for p_t^* in the third equation to obtain

$$p_t = -\frac{\theta\alpha_1}{\beta} + (1-\theta)p_{t-1} + \frac{\theta(1-\alpha_2 a_0)}{\beta}e_t.$$

It follows that

$$p_t - E_{t-1}p_t = \frac{\theta(1-\alpha_2 a_0)}{\beta}e_t.$$

Hence the solution for y_t is

$$\begin{aligned} y_t &= \alpha_1 + \frac{\alpha_2\theta(1-\alpha_2 a_0)}{\beta}e_t + u_t \\ &= \alpha_1 + \frac{\alpha_2\theta(1-\alpha_2 a_0)}{\beta}(\beta m_t + v_t) + (1 - \frac{\alpha_2\theta(1-\alpha_2 a_0)}{\beta})u_t. \end{aligned}$$

(b) (i) If $m_t = \mu + \varepsilon_t$ then an unanticipated shock in period t is a change in ε_t. The effect on p_t is obtained from

$$p_t = -\frac{\theta\alpha_1}{\beta} + (1-\theta)p_{t-1} + \frac{\beta\theta(1-\alpha_2 a_0)}{\beta}\varepsilon_t.$$

(ii) A temporary anticipated shock has the same effect as the solution is backward and not forward looking.

(iii) From the short-run solution for p_t, the long-run solution is obtained from

$$\begin{aligned} p_t &= -\frac{\alpha_1}{\beta} + \frac{1-\alpha_2 a_0}{\beta}e_t \\ &= -\frac{\alpha_1}{\beta} + \frac{1-\alpha_2 a_0}{\beta}(\beta m_t - u_t + v_t) \\ &= -\frac{\alpha_1}{\beta} + \frac{1-\alpha_2 a_0}{\beta}[\beta(\mu + \varepsilon_t) - u_t + v_t], \end{aligned}$$

and is

$$p = -\frac{\alpha_1}{\beta} + (1-\alpha_2 a_0)\mu.$$

In the model, equilibrium implies that $\Delta p_t = 0$, $p_t^* = p_{t-1}$, $E_{t-1}p_t = p_t$, $d_t = y_t$ and $u_t = v_t = \varepsilon_t = 0$. Hence we obtain

$$p = -\frac{\alpha_1}{\beta} + \mu.$$

The solutions are the same if $a_0 = 0$.

(c) The ineffectiveness referred to in the Lucas-Sargent proposition is the effect of money on output. The solution above shows that money affects output in the short run but not the long run.

13.4. Consider the following model of the economy:

$$\begin{aligned} x_t &= -\beta(R_t - E_t\pi_{t+1} - r) \\ \pi_t &= E_t\pi_{t+1} + \alpha x_t + e_t \\ R_t &= \gamma(E_t\pi_{t+1} - \pi^*), \end{aligned}$$

where π_t is inflation, π^* is target inflation, x_t is the output gap, R_t is the nominal interest rate and e_t is a mean zero serially independent shock.

(a) Why is the interest rate equation misspecified?

(b) Correct the specification and state the long-run solution.

(c) What are the short-run solutions for π_t, x_t and R_t?

(d) In the correctly specified model how would the behavior of inflation, output and monetary policy be affected by

(i) a temporary shock e_t

(ii) an expected shock e_{t+1}?

(e) Suppose that the output equation is modified to

$$x_t = -\beta(R_t - E_t\pi_{t+1} - r) - \theta e_t,$$

where e_t can be interpreted as a supply shock. How would the behavior of inflation, output and monetary policy be affected by a supply shock?

Solution

(a) The interest rate equation is misspecified unless the real interest rate r and target inflation sum to zero. Otherwise it should be

$$R_t = r + \pi^* + \gamma(E_t\pi_{t+1} - \pi^*).$$

(b) The long-run solutions are then $x = 0$, $\pi = \pi^*$ and $R = r + \pi^*$.

(c) Solving the model for π_t we obtain

$$\pi_t = [1 - \alpha\beta(\gamma - 1)]E_t\pi_{t+1} + \alpha\beta(\gamma - 1)\pi^* + e_t.$$

This has a unique forward-looking solution if $\gamma > 1$ and is

$$\begin{aligned} \pi_t &= \pi^* + \Sigma_{s=0}^{\infty}[1 - \alpha\beta(\gamma - 1)]^{-s}E_t e_{t+s} \\ &= \pi^* + e_t. \end{aligned}$$

As $E_t\pi_{t+1} = \pi^*$, the short-run solutions for x_t and R_t are

$$\begin{aligned} x_t &= 0 \\ R_t &= r + \pi^*. \end{aligned}$$

(d) (i) In period t inflation would increase by the full amount of the temporary shock e_t, but output and the nominal interest rate would be unaffected.

(ii) If $E_t e_{t+1} = e_{t+1}$ then

$$\begin{aligned} \pi_t &= \pi^* + [1 - \alpha\beta(\gamma - 1)]^{-1}e_{t+1} \\ x_t &= -\beta(\gamma - 1)[1 - \alpha\beta(\gamma - 1)]^{-1}e_{t+1} \\ R_t &= r + \pi^* + \gamma[1 - \alpha\beta(\gamma - 1)]^{-1}e_{t+1}. \end{aligned}$$

(e) The shock e_t now raises inflation and reduces output, which is characteristic of a supply shock. The short-run solutions are now

$$\begin{aligned}
\pi_t &= \pi^* + \Sigma_{s=0}^{\infty}(1-\alpha\theta)[1-\alpha\beta(\gamma-1)]^{-s}E_te_{t+s} \\
&= \pi^* + (1-\alpha\theta)e_t \\
x_t &= -\theta e_t \\
R_t &= r + \pi^*.
\end{aligned}$$

Hence the effect of a temporary shock on inflation is less but now output falls. Monetary policy is still unaffected by the shock. This is because it responds to expected future, and not current, inflation.

13.5 Consider the following New Keynesian model:

$$\begin{aligned}
\pi_t &= \phi + \beta E_t\pi_{t+1} + \gamma x_t + e_{\pi t} \\
x_t &= E_tx_{t+1} - \alpha(R_t - E_t\pi_{t+1} - \theta) + e_{xt} \\
R_t &= \theta + \pi^* + \mu(\pi_t - \pi^*) + \upsilon x_t + e_{Rt},
\end{aligned}$$

where π_t is inflation, π^* is target inflation, x_t is the output gap, R_t is the nominal interest rate, $e_{\pi t}, e_{xt}$ and e_{Rt} are independent, zero-mean iid processes and $\phi = (1-\beta)\pi^*$.

(a) What is the long-run solution?

(b) Write the model in matrix form and obtain the short-run solutions for inflation and the output gap when $\dot{\mu} > 1$ and $\mu < 1$.

(c) Assuming the shocks are uncorrelated, derive the variance of inflation in each case and comment on how the choice of μ and ν affects the variance of inflation

(d) Hence comment on how to tell whether the "great moderation" of inflation in the early 2000's was due to good policy or to good fortune.

Solution

(a) The long-run solution is $\pi = \pi^*$, $x = 0$ and $R = \theta + \pi^*$.

(b) The model written in matrix form is

$$\begin{bmatrix} 1-\beta L^{-1} & -\gamma & 0 \\ -\alpha L^{-1} & 1-L^{-1} & \alpha \\ -\mu & -\nu & 1 \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \\ R_t \end{bmatrix} = \begin{bmatrix} \phi \\ \alpha\theta \\ \theta + \pi^*(1-\mu) \end{bmatrix} + \begin{bmatrix} e_{\pi t} \\ e_{xt} \\ e_{Rt} \end{bmatrix}.$$

This has the general structure

$$B(L)z_t = \eta + \xi_t.$$

The inverse of $B(L)$ is

$$B(L)^{-1} = \frac{1}{f(L)L^{-2}} \begin{bmatrix} 1+\alpha\nu - L^{-1} & \gamma & -\alpha\gamma \\ -\alpha(\mu - L^{-1}) & 1-\beta L^{-1} & -\alpha(1-\beta L^{-1}) \\ \mu + (\alpha\nu - \mu)L^{-1} & (\nu + \mu\gamma) + \beta\nu L^{-1} & 1 + (1+\beta+\alpha\gamma)L^{-1} + \beta L^{-2} \end{bmatrix},$$

where the determinant of $B(L)$ is

$$f(L)L^{-2} = 1 + \alpha(\nu + \mu\gamma) - [1 + \beta(1+\alpha\upsilon) + \alpha\gamma]L^{-1} + \beta L^{-2}.$$

Premultiplying the matrix equation by the adjoint of $B(L)$ gives the following equation for π_t

$$[1 + \alpha(\upsilon + \mu\gamma)]\pi_t - [1 + \beta(1+\alpha\upsilon) + \alpha\gamma]E_t\pi_{t+1} + \beta E_t\pi_{t+2} = \upsilon_t,$$

where

$$\upsilon_t = \alpha(\nu\phi - \gamma\pi^*) + (1+\alpha\upsilon)e_{\pi t} + \gamma e_{xt} - \alpha\gamma e_{Rt}$$

and $L^{-1}e_{\pi t} = E_t e_{\pi,t+1} = 0$, etc.

Writing the solution for π_t as $\pi_t = a + A(L)\varepsilon_t$, and υ_t as $\upsilon_t = b + \phi(L)\varepsilon_t$, the equation for inflation can be re-written as

$$\{[1 + \alpha(\upsilon + \mu\gamma)]A(L) - [1 + \beta(1+\alpha\upsilon) + \alpha\gamma]L^{-1}[A(L) - a_0]$$

$$+\beta L^{-2}[A(L) - a_0 - a_1 L]\}\varepsilon_t = \phi(L)\varepsilon_t$$

where $b = \alpha\pi^*[\nu(1-\beta) + \gamma(\mu - 1)]$. Hence

$$A(L) = \frac{[\beta - (1 + \beta(1 + \alpha\upsilon) + \alpha\gamma)L]a_0 + \beta La_1 + L^2\phi(L)}{\beta - [1 + \beta(1 + \alpha\upsilon) + \alpha\gamma]L + [1 + \alpha(\upsilon + \mu\gamma)]L^2},$$

and the characteristic equation is

$$f(L) = \beta - [1 + \beta(1 + \alpha\upsilon) + \alpha\gamma]L + [1 + \alpha(\upsilon + \mu\gamma)]L^2 = 0.$$

In general, there are two cases to consider: $f(1) = \alpha[\nu(1-\beta) + \gamma(\mu - 1)] \gtrless 0$. These two cases largely reflect whether $\mu \gtrless 1$. If $\mu > 1$ (the Taylor rule sets $\mu = 1.5$) monetary policy responds strongly to inflation; but if $\mu < 1$ then the monetary response is weak.

Case 1: $f(1) = \alpha[\nu(1-\beta) + \gamma(\mu - 1)] > 0$

We may assume this is because $\mu > 1$. In this case both roots are either stable or unstable. Consider the product of the roots of $f(L) = 0$ which, normalizing the coefficient of L^2, is $\frac{\beta}{1+\alpha(\upsilon+\mu\gamma)}$. As $1 > \frac{\beta}{1+\alpha(\upsilon+\mu\gamma)} > 0$ the product of the roots is less than unity and positive. Hence they are unstable. We denote the roots by $0 < \eta_1, \eta_2 < 1$.

Using the method of residues gives for $i = 1, 2$

$$\begin{aligned}
\lim_{L\to\eta_i} f(L)A(L) &= \lim_{L\to\eta_i} [1 + \alpha(\upsilon + \mu\gamma)](L - \eta_1)(L - \eta_2)A(L) \\
&= [\beta - (1 + \beta(1 + \alpha\upsilon) + \alpha\gamma)\eta_i]a_0 + \beta\eta_i a_1 + \eta_i^2\phi(\eta_i) \\
&= 0.
\end{aligned}$$

This gives two equations in the two unknowns a_0 and a_1. Hence a_0 and a_1 are uniquely determined.

The solution for π_t can therefore be obtained as

$$\begin{aligned}
\pi_t &= \frac{1}{[1 + \alpha(\upsilon + \mu\gamma)]L^{-2}(L - \eta_1)(L - \eta_2)}\upsilon_t \\
&= \frac{1}{[1 + \alpha(\upsilon + \mu\gamma)](1 - \eta_1 L^{-1})(1 - \eta_2 L^{-1})}\upsilon_t \\
&= \frac{1}{[1 + \alpha(\upsilon + \mu\gamma)](\eta_1 - \eta_2)}[\frac{\eta_1}{1 - \eta_1 L^{-1}} - \frac{\eta_2}{1 - \eta_2 L^{-1}}]\upsilon_t \\
&= \frac{1}{1 + \alpha(\upsilon + \mu\gamma)}[\frac{\eta_1}{\eta_1 - \eta_2}\Sigma_{s=0}^{\infty}\eta_1^s E_t\upsilon_{t+s} - \frac{\eta_2}{\eta_1 - \eta_2}\Sigma_{s=0}^{\infty}\eta_2^s E_t\upsilon_{t+s}].
\end{aligned}$$

The solution uses the value of v_t. This is the first element of the right-hand side of the matrix solution. For π_t the solution is

$$\pi_t = \pi^* + \frac{1}{1+\alpha(\upsilon+\mu\gamma)}[(1+\alpha\upsilon)e_{\pi t} + \gamma e_{xt} - \alpha\gamma e_{Rt}].$$

Thus, average inflation equals the target rate. Hence $a = \pi^*$. Inflation deviates from target due to the three shocks. Positive inflation and output shocks cause inflation to rise above target, but positive interest rate shocks cause inflation to fall below target. We note that a forward-looking Taylor rule in which $E_t\pi_{t+1}$ replaces π_t and $E_t x_{t+1}$ replaces x_t gives a similar result.

The basic solution for the output gap is the same as for π_t. The difference is in the definition of v_t. (Or substitute for R_t from the policy rule and for $E_t\pi_{t+1} = \pi^*$.) Consequently,

$$\begin{aligned}
x_t &= E_t x_{t+1} - \alpha(R_t - \pi^* - \theta) + e_{xt} \\
&= E_t x_{t+1} - \alpha\mu(\pi_t - \pi^*) - \alpha\mu\upsilon x_t + e_{xt} - \alpha e_{Rt} \\
&= E_t x_{t+1} - \frac{\alpha\mu}{1+\alpha(\upsilon+\mu\gamma)}[(1+\alpha\upsilon)e_{\pi t} + \gamma e_{xt} - \alpha\gamma e_{Rt}] - \alpha\mu\upsilon x_t + e_{xt} - \alpha e_{Rt} \\
&= \frac{1}{1+\alpha\mu\upsilon}E_t x_{t+1} - \frac{1}{1+\alpha(\upsilon+\mu\gamma)}(\alpha\mu e_{\pi t} + e_{xt} - e_{Rt}).
\end{aligned}$$

Solving forwards gives the solution for x_t as

$$x_t = -\frac{1}{1+\alpha(\upsilon+\mu\gamma)}(\alpha\mu e_{\pi t} + e_{xt} - e_{Rt}).$$

Hence the expected output gap is zero.

Case 2: $f(1) = \alpha[\nu(1-\beta) + \gamma(\mu-1)] < 0$

This is probably because $\mu < 1$. It implies a saddlepath solution with the roots of $f(L) = 0$ on each side of unity. Let $\eta_1 \geq 1$ and $\eta_2 < 1$. Applying the method of residuals, we have one restriction associated with η_2 which is given by

$$\begin{aligned}
\lim_{L\to\eta_2} f(L)A(L) &= \lim_{L\to\eta_2}(L-\eta_1)(L-\eta_2)A(L) \\
&= [\beta - (1+\beta(1+\alpha\upsilon)+\alpha\gamma)\eta_2]a_0 + \beta\eta_2 a_1 + \eta_2^2\phi(\eta_2) \\
&= 0
\end{aligned}$$

It follows that we have only one equation and two unknowns: a_0, a_1. The solution is therefore indeterminate.

Noting that

$$f(L)A(L) = [\beta - (1 + \beta(1 + \alpha\upsilon) + \alpha\gamma)L]a_0 + \beta La_1 + L^2\phi(L),$$

and that

$$\begin{aligned} f(L) &= \beta - [1 + \beta(1 + \alpha\upsilon) + \alpha\gamma]L + [1 + \alpha(\upsilon + \mu\gamma)]L^2 \\ &= [1 + \alpha(\upsilon + \mu\gamma)](L - \eta_1)(L - \eta_2) \\ &= -[1 + \alpha(\upsilon + \mu\gamma)]\eta_1 L(1 - \frac{1}{\eta_1}L)(1 - \eta_2 L^{-1}), \end{aligned}$$

it follows that

$$(1 - \frac{1}{\eta_1}L)A(L) = -\frac{[\beta - (1 + \beta(1 + \alpha\upsilon) + \alpha\gamma)L]a_0 + \beta La_1 + L^2\phi(L)}{[1 + \alpha(\upsilon + \mu\gamma)]\eta_1 L(1 - \eta_2 L^{-1})}.$$

Subtracting $[\beta - (1 + \beta(1 + \alpha\upsilon) + \alpha\gamma)\eta_2]a_0 + \beta\eta_2 a_1 + \eta_2^2\phi(\eta_2) = 0$ from the numerator and simplifying gives

$$\begin{aligned} (1 - \frac{1}{\eta_1}L)A(L)\varepsilon_t &= -\frac{[a_0\alpha\gamma(L - \eta_2) + a_1\beta(L - \eta_2) + L^2\phi(L) - \eta_2^2\phi(\eta_2)}{[1 + \alpha(\upsilon + \mu\gamma)]\eta_1 L(1 - \eta_2 L^{-1})}\varepsilon_t \\ &= -\frac{1}{[1 + \alpha(\upsilon + \mu\gamma)]\eta_1}[a_0\alpha\gamma + a_1\beta + \frac{L^2\phi(L) - \eta_2^2\phi(\eta_2)}{L(1 - \eta_2 L^{-1})}]\varepsilon_t \\ &= -\frac{1}{[1 + \alpha(\upsilon + \mu\gamma)]\eta_1}[a_0\alpha\gamma + a_1\beta + \frac{\eta_2[\eta_2^{-1}L - 1 + 1 - \eta_2\phi(\eta_2)L^{-1}\phi(L)^{-1}]\phi(L)}{1 - \eta_2 L^{-1}}]\varepsilon_t \\ &= -\frac{1}{[1 + \alpha(\upsilon + \mu\gamma)]\eta_1}[(a_0\alpha\gamma + a_1\beta)\varepsilon_t - \upsilon_t + \eta_2\Sigma_{s=0}^{\infty}\eta_2^s E_t\upsilon_{t+s}]. \end{aligned}$$

As $\pi_t = a + A(L)\varepsilon_t$,

$$\begin{aligned} \upsilon_t &= b + \phi(L)\varepsilon_t \\ b &= \alpha(\nu\phi - \gamma\pi^*) = \alpha\pi^*[\nu(1 - \beta) - \gamma] \\ \phi(L)\varepsilon_t &= (1 + \alpha\upsilon)e_{\pi t} + \gamma e_{xt} - \alpha\gamma e_{Rt}, \end{aligned}$$

the solution for π_t is

$$\begin{aligned}\pi_t - a &= \frac{1}{\eta_1}(\pi_{t-1} - a) - \frac{1}{[1+\alpha(\upsilon+\mu\gamma)]\eta_1}[(a_0\alpha\gamma + a_1\beta)\varepsilon_t - (v_t - b) + \eta_2\Sigma_{s=0}^{\infty}\eta_2^s E_t(v_{t+s} - b)] \\ &= \frac{1}{\eta_1}(\pi_{t-1} - a) + \frac{1 - a_0\alpha\gamma - a_1\beta - \eta_2}{[1+\alpha(\upsilon+\mu\gamma)]\eta_1}[(1+\alpha\upsilon)e_{\pi t} + \gamma e_{xt} - \alpha\gamma e_{Rt}].\end{aligned}$$

Thus, under the rule, the effect of shocks on π_t is indeterminate. It can also be shown that

$$E\pi_t = a = \frac{\phi}{1-\beta} = \pi^*.$$

Hence π_t is generated by the AR(1) process

$$\pi_t - \pi^* = \frac{1}{\eta_1}(\pi_{t-1} - \pi^*) + \frac{1 - a_0\alpha\gamma - a_1\beta - \eta_2}{[1+\alpha(\upsilon+\mu\gamma)]\eta_1}[(1+\alpha\upsilon)e_{\pi t} + \gamma e_{xt} - \alpha\gamma e_{Rt}].$$

Thus, in this case, inflation is more persistent than in the previous case.

(c) If the shocks are uncorrelated the variance of inflation in the two cases are

Case 1:

$$Var_t(\pi_t) = \frac{1}{[1+\alpha(\upsilon+\mu\gamma)]^2}[(1+\alpha\upsilon)^2\sigma_\pi^2 + \gamma^2\sigma_x^2 + (\alpha\gamma)^2\sigma_R^2],$$

where σ_i^2 $(i = \pi, x, R)$ are the variances of the shocks.

Case 2:

$$Var(\pi_t) = \frac{1}{1-\eta_1^{-2}}\left[\frac{1 - a_0\alpha\gamma - a_1\beta - \eta_2}{[1+\alpha(\upsilon+\mu\gamma)]\eta_1}\right]^2[(1+\alpha\upsilon)^2\sigma_\pi^2 + \gamma^2\sigma_x^2 + (\alpha\gamma)^2\sigma_R^2].$$

In Case 1, as $\mu \to \infty$, the variance of inflation goes to zero; and, for any finite value of μ, as $\nu \to 0$ the variance of inflation decreases. This suggests that if the policy objective is to minimise the variation of inflation about the target level π^* - which is equivalent to minimising the variance of inflation as $E\pi_t = \pi^*$ - then strict inflation targeting ($\nu = 0$) is preferable to flexible inflation targeting ($\nu > 0$).

(d) There has been a debate about whether the "great moderation" of the 2000's was due to good policy or good luck. Our results indicate that good policy requires that $\mu > 1$ and that there are no shocks to the policy equation so that $\sigma_R^2 = 0$. Good luck is when the variances of inflation and the output gap (σ_π^2 and σ_x^2) are small.

We also recall that in this model another advantage of a strong monetary policy response to inflation is that inflation is not then persistent. In other words, inflation responds quickly to monetary policy.

13.6. Consider the following model of Broadbent and Barro (1997):

$$
\begin{aligned}
y_t &= \alpha(p_t - E_{t-1}p_t) + e_t \\
d_t &= -\beta r_t + \varepsilon_t \\
m_t &= y_t + p_t - \lambda R_t \\
r_t &= R_t - E_t \Delta p_{t+1} \\
y_t &= d_t,
\end{aligned}
$$

where e_t and ε_t are zero-mean mutually and serially correlated shocks.

(a) Derive the solution to the model

(i) under money supply targeting,

(ii) inflation targeting,

(b) Derive the optimal money supply rule if monetary policy minimizes $E_t(p_{t+1} - E_t p_{t+1})^2$ subject to the model of the economy.

(c) What does this policy imply for inflation and the nominal interest rate?

(d) Derive the optimal interest rate rule.

(e) How would these optimal policies differ if monetary policy was based on targeting inflation instead of the price level?

Solution

(a) (i) Money supply targeting

The money supply m_t is exogenous. The model can be reduced to one equation for the price level

$$p_t = \frac{1}{\beta(1+\lambda)+\alpha(\beta+\lambda)}[\beta\lambda E_t p_{t+1} + \alpha(\beta+\lambda)E_{t-1}p_t - \beta m_t + (\beta+\lambda)e_t - \lambda\varepsilon_t]$$

If $p_t = A(L)\xi_t$ where $\xi_t = -\beta m_t + (\beta+\lambda)e_t - \lambda\varepsilon_t$ then the equation may be re-written as

$$A(L)\xi_t = \frac{1}{\beta(1+\lambda)+\alpha(\beta+\lambda)}\{\beta\lambda[A(L)-a_0)L^{-1}] + \alpha(\beta+\lambda)[A(L)-a_0]+1\}\xi_t$$

Hence

$$A(L) = \frac{\beta\lambda - [1-a_0\alpha(\beta+\lambda)]L}{\beta\lambda - \beta(1+\lambda)L}$$

Since the root $L = \frac{\lambda}{1+\lambda} < 1$ we have an unstable solution. Using the method of residues

$$\lim_{L\to\frac{\lambda}{1+\lambda}}[\beta\lambda - \beta(1+\lambda)L]A(L) = \beta\lambda - [1-a_0\alpha(\beta+\lambda)]\frac{\lambda}{1+\lambda} = 0$$

hence

$$a_0 = \frac{1-\beta(1+\lambda)}{\alpha(\beta+\lambda)}$$

It then follows that $A(L) = 1$. Thus the solution is

$$p_t = -\beta m_t + (\beta+\lambda)e_t - \lambda\varepsilon_t.$$

(ii) Inflation targeting

The interest rate R_t is now exogenous. The money demand equation can be ignored and the rest of the model can be reduced to

$$p_t = \frac{\beta}{\alpha}E_t p_{t+1} + E_{t-1}p_t - \frac{\beta}{\alpha}R_t - \frac{1}{\alpha}e_t + \frac{1}{\alpha}\varepsilon_t.$$

If $p_t = A(L)\xi_t$ where $\xi_t = -\frac{\beta}{\alpha}R_t - \frac{1}{\alpha}e_t + \frac{1}{\alpha}\varepsilon_t$, then the equation may be re-written as

$$A(L)\xi_t = \{\frac{\beta}{\alpha}[A(L)-a_0)L^{-1}] + [A(L)-a_0]+1\}\xi_t.$$

Hence

$$A(L) = a_0 - \frac{1-a_0}{\beta}L.$$

The solution is therefore

$$p_t = a_0 \xi_t - \frac{1-a_0}{\beta}\xi_{t-1},$$

which is indeterminate.

(b) The optimal money supply rule is obtained by minimizing $E_t(p_{t+1} - E_t p_{t+1})^2$ subject to $p_t = -\beta m_t + (\beta + \lambda)e_t - \lambda\varepsilon_t$. As

$$p_{t+1} - E_t p_{t+1} = \beta(m_{t+1} - E_t m_{t+1}) + (\beta + \lambda)e_{t+1} - \lambda\varepsilon_{t+1},$$

optimal monetary policy would be to offset the shocks so that

$$m_{t+1} - E_t m_{t+1} = -\frac{\beta+\lambda}{\beta}e_{t+1} + \frac{\lambda}{\beta}\varepsilon_{t+1}.$$

However, as these shocks are unknown at time t, the optimal monetary policy is to keep the money supply constant when $E_t(p_{t+1} - E_t p_{t+1}) = 0$. If the shocks are uncorrelated the conditional variance of inflation is

$$E_t(p_{t+1} - E_t p_{t+1})^2 = \left(\frac{\beta+\lambda}{\beta}\right)^2 \sigma_e^2 + \left(\frac{\lambda}{\beta}\right)^2 \sigma_\varepsilon^2.$$

(c) Inflation is then

$$p_t = -\beta m + (\beta + \lambda)e_t - \lambda\varepsilon_t$$

and the interest rate is

$$\begin{aligned} R_t &= E_t \Delta p_{t+1} - \frac{\alpha}{\beta}(p_t - E_{t-1}p_t) - \frac{1}{\beta}e_t + \frac{1}{\beta}\varepsilon_t \\ &= -\frac{1+\alpha(\beta+\lambda)}{\beta}e_t + \frac{1+\alpha\lambda}{\beta}\varepsilon_t. \end{aligned}$$

(d) The optimal interest rate rule is obtained by minimizing $E_t(p_{t+1} - E_t p_{t+1})^2$ subject to

$$\begin{aligned} p_t &= a_0 \xi_t - \frac{1-a_0}{\beta}\xi_{t-1} \\ &= -a_0 \frac{\beta}{\alpha}R_t + \frac{1-a_0}{\alpha}R_{t-1} - \frac{a_0}{\alpha}e_t + \frac{a_0}{\alpha}\varepsilon_t + \frac{1-a_0}{\alpha\beta}e_{t-1} - \frac{1-a_0}{\alpha\beta}\varepsilon_{t-1}. \end{aligned}$$

Hence

$$p_{t+1} - E_t p_{t+1} = -a_0 \frac{\beta}{\alpha}(R_{t+1} - E_t R_{t+1}) - \frac{a_0}{\alpha} e_{t+1} + \frac{a_0}{\alpha} \varepsilon_{t+1}.$$

Optimal monetary policy would be to offset the shocks so that

$$R_{t+1} - E_t R_{t+1} = -\frac{1}{\beta} e_{t+1} + \frac{1}{\beta} \varepsilon_{t+1}.$$

As these shocks are unknown at time t, if the shocks are uncorrelated, the optimal monetary policy is to keep the nominal interest rate constant when

$$E_t(p_{t+1} - E_t p_{t+1})^2 = \left(\frac{a_0}{\alpha}\right)^2 (\sigma_e^2 + \sigma_\varepsilon^2).$$

(d) Under inflation targeting the equivalent objective function would be

$$E_t(\Delta p_{t+1} - E_t \Delta p_{t+1})^2 = E_t(p_{t+1} - E_t p_{t+1})^2$$

and so policy would be unchanged.

13.7. Suppose that a monetary authority is a strict inflation targeter attempting to minimize $E(\pi_t - \pi^*)^2$ subject to the following model of the economy

$$\pi_t = \alpha_t R_t + z_t + e_t,$$

where $\alpha_t = \alpha + \varepsilon_t$, $E(z_t) = z + \varepsilon_{zt}$ and $\varepsilon_{\alpha t}$ and ε_{zt} are random measurement errors of α and z, respectively; $\varepsilon_{\alpha t}$, ε_{zt} and e_t are mutually and independently distributed random variables with zero means and variances σ_z^2, σ_α^2 and σ_e^2.

(a) What is the optimal monetary policy

(i) in the absence of measurement errors,

(ii) in the presence of measurement errors?

(b) What are broader implications of these results for monetary policy?

Solution

(a) Inflation is determined by

$$\pi_t = (\alpha + \varepsilon_{\alpha t})R_t + z + \varepsilon_{zt} + e_t.$$

(i) In the absence of measurement errors this becomes

$$\pi_t = \alpha R_t + z + e_t.$$

The objective function can then be re-written as

$$\begin{aligned} E(\pi_t - \pi^*)^2 &= E[\alpha R_t + z + e_t - \pi^*]^2 \\ &= [\alpha R_t + z - \pi^*]^2 + \sigma_e^2. \end{aligned}$$

Minimizing this with respect to R_t gives

$$2\alpha(\alpha R_t + z - \pi^*) = 0,$$

or

$$R_t = \frac{1}{\alpha}(\pi^* - z).$$

(ii) In the presence of measurement errors

$$\begin{aligned} E(\pi_t - \pi^*)^2 &= E[(\alpha + \varepsilon_{\alpha t})R_t + z + \varepsilon_{zt} + e_t - \pi^*]^2 \\ &= [\alpha R_t + z - \pi^*]^2 + \sigma_\alpha^2 R_t^2 + \sigma_z^2 + \sigma_e^2. \end{aligned}$$

Minimizing this with respect to R_t gives

$$2\alpha(\alpha R_t + z - \pi^*) + 2\sigma_\alpha^2 R_t = 0.$$

Hence,

$$\begin{aligned} R_t &= \frac{\alpha}{\alpha^2 + \sigma_\alpha^2}(\pi^* - z) \\ &\lessgtr \frac{1}{\alpha}(\pi^* - z) \quad \text{as} \quad \pi^* \gtrless z. \end{aligned}$$

(b) These results show that optimal monetary policy does not respond to either shocks or measurement errors, but only to the mean of z_t. However, the presence of measurement errors in the coefficient of R_t tends to reduce the absolute size of the monetary response.

13.8. A highly stylized model of an open economy is

$$\begin{aligned} p_t &= \alpha p_{t-1} + \theta(s_t - p_t) \\ s_t &= R_t + R_{t+1}, \end{aligned}$$

where p_t is the price level, s_t is the exchange rate and R_t is the nominal interest rate. Suppose that monetary policy aims to choose R_t and R_{t+1} to minimize

$$L = (p_t - p^*)^2 + \beta(p_{t+1} - p^*)^2 + \gamma(R_t - R^*)^2,$$

where $p_{t-1} = R_{t+2} = 0$.

(a) Find the time consistent solutions for R_t and R_{t+1}. (Hint: first find R_{t+1} taking R_t as given.)

(b) Find the optimal solution by optimizing simultaneously with respect to R_t and R_{t+1}.

(c) Compare the two solutions and the significance of γ.

Solution

(a) The price level is determined by

$$\begin{aligned} p_t &= \frac{\alpha}{1+\theta} p_{t-1} + \frac{\theta}{1+\theta}(R_t + R_{t+1}) \\ &= \frac{\theta}{1+\theta}(R_t + R_{t+1}). \end{aligned}$$

The loss function can therefore be re-written as

$$L = [\frac{\theta}{1+\theta}(R_t + R_{t+1}) - p^*]^2 + \beta[\frac{\theta}{1+\theta}R_{t+1} + \frac{\alpha\theta}{(1+\theta)^2}(R_t + R_{t+1}) - p^*]^2 + \gamma(R_t - R^*)^2.$$

Minimizing this with respect to R_{t+1} taking p_t and R_t as given yields

$$\begin{aligned}\frac{\partial L}{\partial R_{t+1}} &= 2\frac{\theta}{1+\theta}[\frac{\theta}{1+\theta}(R_t+R_{t+1})-p^*] \\ &\quad +2\beta\frac{\theta}{1+\theta}(1+\frac{\alpha}{1+\theta})[\frac{\theta}{1+\theta}R_{t+1}+\frac{\alpha\theta}{(1+\theta)^2}(R_t+R_{t+1})-p^*] \\ &= 0.\end{aligned}$$

Hence,

$$\begin{aligned}R_{t+1} &= -\frac{1+\frac{\alpha}{1+\theta}(1+\frac{\alpha}{1+\theta})}{1+(1+\frac{\alpha}{1+\theta})^2}R_t+\frac{2(1+\theta)+\alpha}{1+(1+\frac{\alpha}{1+\theta})^2}p^* \\ &= -\mu R_t+\delta p^*.\end{aligned}$$

Next maximize L with respect to R_t and R_{t+1} subject to this solution for R_{t+1} as a constraint with Lagrange multiplier λ. The first-order conditions are

$$\begin{aligned}\frac{\partial L}{\partial R_{t+1}} &= \lambda+2\frac{\theta}{1+\theta}[\frac{\theta}{1+\theta}(R_t+R_{t+1})-p^*] \\ &\quad +2\beta\frac{\theta}{1+\theta}(1+\frac{\alpha}{1+\theta})[\frac{\theta}{1+\theta}R_{t+1}+\frac{\alpha\theta}{(1+\theta)^2}(R_t+R_{t+1})-p^*] \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial L}{\partial R_t} &= -\lambda\mu+2\frac{\theta}{1+\theta}[\frac{\theta}{1+\theta}(R_t+R_{t+1})-p^*] \\ &\quad +2\beta\frac{\theta}{1+\theta}(1+\frac{\alpha}{1+\theta})[\frac{\alpha\theta}{(1+\theta)^2}(R_t+R_{t+1})-p^*]+2\gamma(R_t-R^*) \\ &= 0.\end{aligned}$$

Hence,

$$\begin{bmatrix} \left(\frac{\theta}{1+\theta}\right)^2[1+(1+\frac{\alpha}{1+\theta})^2] & \left(\frac{\theta}{1+\theta}\right)^2[1+\frac{\alpha}{1+\theta}(1+\frac{\alpha}{1+\theta})] & 1 \\ \left(\frac{\theta}{1+\theta}\right)^2[1+\beta(1+\frac{\alpha}{1+\theta})\frac{\alpha}{1+\theta}] & \gamma+\left(\frac{\theta}{1+\theta}\right)^2[1+\beta(1+\frac{\alpha}{1+\theta})\frac{\alpha}{1+\theta}] & -\mu \\ 1 & -\mu & 0 \end{bmatrix} \begin{bmatrix} R_{t+1} \\ R_t \\ \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2(1+\theta)+\alpha \\ \gamma & \left(\frac{\theta}{1+\theta}\right)^2[1+\beta(1+\frac{\alpha}{1+\theta})\frac{\alpha}{1+\theta}] \\ 0 & \delta \end{bmatrix} \begin{bmatrix} R^* \\ p^* \end{bmatrix}$$

from which R_{t+1} and R_t can be solved.

(b) Maximizing L with respect to R_{t+1} and R_t gives

$$\begin{aligned} \frac{\partial L}{\partial R_{t+1}} &= 2\frac{\theta}{1+\theta}[\frac{\theta}{1+\theta}(R_t+R_{t+1})-p^*] \\ &\quad +2\beta\frac{\theta}{1+\theta}(1+\frac{\alpha}{1+\theta})[\frac{\theta}{1+\theta}R_{t+1}+\frac{\alpha\theta}{(1+\theta)^2}(R_t+R_{t+1})-p^*] \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial L}{\partial R_t} &= 2\frac{\theta}{1+\theta}[\frac{\theta}{1+\theta}(R_t+R_{t+1})-p^*] \\ &\quad +2\beta\frac{\theta}{1+\theta}(1+\frac{\alpha}{1+\theta})[\frac{\alpha\theta}{(1+\theta)^2}(R_t+R_{t+1})-p^*]+2\gamma(R_t-R^*) \\ &= 0. \end{aligned}$$

Hence,

$$\begin{bmatrix} 1+(1+\frac{\alpha}{1+\theta})^2 & 1+\frac{\alpha}{1+\theta}(1+\frac{\alpha}{1+\theta}) \\ \left(\frac{\theta}{1+\theta}\right)^2[1+\beta(1+\frac{\alpha}{1+\theta})\frac{\alpha}{1+\theta}] & \gamma+\left(\frac{\theta}{1+\theta}\right)^2[1+\beta(1+\frac{\alpha}{1+\theta})\frac{\alpha}{1+\theta}] \end{bmatrix} \begin{bmatrix} R_{t+1} \\ R_t \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2(1+\theta)+\alpha \\ \gamma & \left(\frac{\theta}{1+\theta}\right)^2[1+\beta(1+\frac{\alpha}{1+\theta})\frac{\alpha}{1+\theta}] \end{bmatrix} \begin{bmatrix} R^* \\ p^* \end{bmatrix}.$$

From which R_{t+1} and R_t can be solved.

(c) Due to the constraint the two solutions will be different. Only if $\lambda = 0$ - i.e. the constraint is ignored - are they the same. It follows that the second solution is not time consistent.

13.9. Consider the following model of an open economy:

$$\begin{aligned} \pi_t &= \mu + \beta E_t \pi_{t+1} + \gamma x_t + e_{\pi t} \\ x_t &= -\alpha(R_t - E_t\pi_{t+1} - \theta) + \phi(s_t + p_t^* - p_t) + e_{xt} \\ \Delta s_{t+1} &= R_t - R_t^* + e_{st}, \end{aligned}$$

where $e_{\pi t}$, e_{xt} and e_{st} are mean zero mutually and serially independent shocks to inflation, output and the exchange rate.

(a) Derive the long-run solution making any additional assumptions thought necessary.

(b) Derive the short-run solution for inflation

(c) Each period monetary policy is set to minimize $E_t(\pi_{t+1} - \pi^*)^2$, where π^* is the long-run solution for π, on the assumption that the interest rate chosen will remain unaltered indefinitely and the foreign interest rate and price level will remain unchanged. Find the optimal value of R_t.

Solution

(a) Assuming there is a unique solution, and that in the long-run PPP holds, the long-run solution is

$$\begin{aligned}
p &= s + p^* \\
\pi &= \frac{\mu + \alpha\gamma\theta}{1 - \alpha\gamma - \beta} - \frac{\alpha\gamma}{1 - \alpha\gamma - \beta} R^* \\
x &= -\alpha(R^* - \pi - \theta).
\end{aligned}$$

In order for the long-run output gap to be zero we need the additional assumption that $\pi = R^* - \theta$. It then follows that $\pi = \frac{\mu}{1-\beta}$.

(b) The model may be written in matrix form as

$$\begin{bmatrix} 1 - L - \beta(L^{-1} - 1) & -\gamma & 0 \\ \phi + \alpha(L^{-1} - 1) & 1 & -\phi \\ 0 & 0 & L^{-1} - 1 \end{bmatrix} \begin{bmatrix} p_t \\ x_t \\ s_t \end{bmatrix} = \begin{bmatrix} \mu \\ -\alpha(R_t - \theta) + \phi p_t^* \\ R_t - R_t^* \end{bmatrix} + \begin{bmatrix} e_{\pi t} \\ e_{xt} \\ e_{st} \end{bmatrix},$$

or

$$A(L)z_t = w_t + e_t.$$

The inverse of $A(L)$ is

$$A(L)^{-1} = [\det A(L)]^{-1} B(L),$$

where

$$B(L) = \begin{bmatrix} L^{-1} - 1 & \gamma(L^{-1} - 1) & \gamma\phi \\ (L^{-1} - 1)(\phi - \alpha + \alpha L^{-1}) & (L^{-1} - 1)[1 - \gamma(\phi - \alpha + \alpha L^{-1})] & -\phi(1 + \beta - L - \beta L^{-1}) \\ 0 & 0 & 1 + \beta + \gamma(\phi - \alpha) - L + (\alpha\gamma - \beta)L^{-1} \end{bmatrix}$$

and

$$\begin{aligned}\det A(L) &= (L^{-1}-1)(1+\beta+\gamma(\phi-\alpha)-L+(\alpha\gamma-\beta)L^{-1}) \\ &= -L^{-2}(1-L)\{L^2-[1+\beta+\gamma(\phi-\alpha)]L+(\beta-\alpha\gamma)\} \\ &= -L^{-2}(1-L)f(L).\end{aligned}$$

As $f(1)=-\gamma\phi$, one of its roots is greater than unity and one less than unity. Hence we may write the roots of $(1-L)f(L)$ as $\lambda_1>1$, $\lambda_2<1$ and 1. It follows that the solution to the model is

$$(1-L)(1-\lambda_1^{-1}L)(1-\lambda_2L^{-1})z_t=\lambda_1^{-1}LB(L)(w_t+e_t).$$

The solution for inflation is therefore

$$\begin{aligned}\pi_t &= \lambda_1^{-1}\pi_{t-1}+\lambda_1^{-1}E_t\Sigma_{j=0}^{\infty}\lambda_2^j[-\alpha\gamma R_{t+j}+\gamma(\alpha+\phi)R_{t+j-1}-\gamma\phi R^*_{t+j-1}+\gamma\phi\Delta p^*_{t+j} \\ &\quad +\Delta e_{\pi,t+j}+\gamma\Delta e_{x,t+j}+\gamma\phi e_{s,t+j-1}] \\ &= \lambda_1^{-1}\pi_{t-1}+\lambda_1^{-1}E_t\Sigma_{j=0}^{\infty}\lambda_2^j\{-[\alpha\gamma-\gamma(\alpha+\phi)\lambda_2]R_{t+j}-\gamma\phi R^*_{t+j-1} \\ &\quad +\gamma\phi\Delta p^*_{t+j}\}+\lambda_1^{-1}\gamma(\alpha+\phi)R_{t-1}+\lambda_1^{-1}(1-\lambda_2)e_{\pi t}-\lambda_1^{-1}e_{\pi,t-1} \\ &\quad +\lambda_1^{-1}\gamma(1-\lambda_2)e_{xt}-\lambda_1^{-1}\gamma e_{x,t-1}+\lambda_1^{-1}\gamma\phi\lambda_2e_{s,t}+\lambda_1^{-1}\gamma\phi e_{s,t-1}.\end{aligned}$$

(c) If $E_tR_{t+j}=R_t$, then inflation is generated by an equation of the form

$$\begin{aligned}\pi_t &= \lambda_1^{-1}\pi_{t-1}-\lambda_1^{-1}(1-\lambda_2)^{-1}[\alpha\gamma-\gamma(\alpha+\phi)\lambda_2]R_t+\lambda_1^{-1}\gamma(\alpha+\phi)R_{t-1} \\ &\quad +\lambda_1^{-1}E_t\Sigma_{j=0}^{\infty}\lambda_2^jz_{t+j}+u_t \\ &= \lambda_1^{-1}\pi_{t-1}-\eta R_t+\nu R_{t-1}+q_t+u_t,\end{aligned}$$

where

$$\begin{aligned}q_t &= \lambda_1^{-1}E_t\Sigma_{j=0}^{\infty}\lambda_2^jz_{t+j} \\ z_t &= -\gamma\phi R^*_{t-1}+\gamma\phi\Delta p^*_t \\ u_t &= \lambda_1^{-1}(1-\lambda_2)e_{\pi t}-\lambda_1^{-1}e_{\pi,t-1}+\lambda_1^{-1}\gamma(1-\lambda_2)e_{xt}-\lambda_1^{-1}\gamma e_{x,t-1}+\lambda_1^{-1}\gamma\phi\lambda_2e_{s,t}+\lambda_1^{-1}\gamma\phi e_{s,t-1}.\end{aligned}$$

Hence, if q_{t+1} is treated as a constant q,

$$\begin{aligned}\pi_{t+1} &= \lambda_1^{-1}\pi_t - \eta R_{t+1} + \nu R_t + q_{t+1} + u_{t+1} \\ &= \lambda_1^{-1}\pi_t - (\eta - \nu)R_t + q + u_{t+1} \\ &= w_t + u_{t+1},\end{aligned}$$

where

$$w_t = \lambda_1^{-1}\pi_t - (\eta - \nu)R_t + q.$$

It follows that

$$E_t(\pi_{t+1} - \pi^*)^2 = (w_t - \pi^*)^2 + \sigma_u^2 - 2\lambda_1^{-1}E_t\{\pi_t[-\lambda_1^{-1}e_{\pi t} - \lambda_1^{-1}\gamma e_{xt} + \lambda_1^{-1}\gamma\phi e_{st}]\}.$$

As the last two terms do not involve R_t, the optimal value of R_t is

$$R_t = \frac{\lambda_1^{-1}\pi_t + q - \pi^*}{(\eta - \nu)}$$

Thus, an increase in current inflation, or in current and expected future foreign interest rates, or the foreign price level, would require a higher domestic interest rate, but temporary shocks would have no effect.

Chapter 14

14.1. Consider a variant on the basic real business cycle model. The economy is assumed to maximize $E_t \sum_{s=0}^{\infty} \beta^s \frac{{c_{t+s}}^{1-\sigma}}{1-\sigma}$ subject to

$$\begin{aligned} y_t &= c_t + i_t \\ y_t &= A_t k_t^{\alpha} \\ \Delta k_{t+1} &= i_t - \delta k_t \\ \ln A_t &= \rho \ln A_{t-1} + e_t, \end{aligned}$$

where y_t is output, c_t is consumption, i_t is investment, k_t is the capital stock, A_t is technical progress and $e_t \sim i.i.d(0, \omega^2)$.

(a) Derive

(i) the optimal short-run solution,

(ii) the steady-state solution,

(ii) a log-linearization to the short-run solution about its steady state in $\ln c_t - \ln c$ and $\ln k_t - \ln k$, where $\ln c$ and $\ln k$ are the steady-state values of $\ln c_t$ and $\ln k_t$.

(b) If, in practice, output, consumption and capital are non-stationary I(1) variables,

(i) comment on why this model is not a useful specification.

(ii) Suggest a simple re-specification of the model that would improve its usefulness.

(c) In practice, output, consumption and capital also have independent sources of random variation.

(i) Why is this not compatible with this model?

(ii) Suggest possible ways in which the model might be re-specified to achieve this.

Solution

(a) (i) The economy's resource constraint is

$$A_t k_t^{\alpha} = c_t + k_{t+1} - (1 - \delta) k_t.$$

As technological change makes the problem stochastic, we maximize the value function

$$\begin{aligned} V_t &= U(c_t) + \beta E_t(V_{t+1}) \\ &= \frac{{c_t}^{1-\sigma}}{1-\sigma} + \beta E_t(V_{t+1}), \end{aligned}$$

subject to the resource constraint. The first-order conditions for this stochastic dynamic programming problem are

$$\frac{\partial V_t}{\partial c_t} = c_t^{-\sigma} + \beta E_t[\frac{\partial V_{t+1}}{\partial c_{t+1}} \cdot \frac{\partial c_{t+1}}{\partial c_t}] = 0.$$

Noting that

$$V_{t+1} = U(C_{t+1}) + \beta E_{t+1}(V_{t+2}),$$

and hence

$$\frac{\partial V_{t+1}}{\partial c_{t+1}} = c_{t+1}^{-\sigma},$$

and that

$$\frac{\partial c_{t+1}}{\partial c_t} = \frac{\frac{\partial c_{t+1}}{\partial k_{t+1}}}{\frac{\partial c_t}{\partial k_{t+1}}},$$

from the budget constraints for periods t and $t+1$, we can show that

$$\frac{\partial c_{t+1}}{\partial c_t} = \alpha A_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta.$$

Hence

$$\frac{\partial V_t}{\partial c_t} = c_t^{-\sigma} + \beta E_t[c_{t+1}^{-\sigma} \cdot \alpha A_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta] = 0.$$

This gives the Euler equation

$$E_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} (\alpha A_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta) \right] = 1.$$

The optimal short-run solution to the model is the Euler equation plus the economy's resource constraint.

In deriving the solution to the model it is usual in RBC analysis to invoke certainty equivalence. This allows all random variables to be replaced by their conditional expectations. In the Euler equation, strictly we should take account of the conditional covariance terms involving c_{t+1}, k_{t+1} and Z_{t+1} and, in particular, $cov_t(c_{t+1}, k_{t+1})$.

(ii) The steady-state solution - assuming it exists - satisfies $\Delta c_{t+1} = \Delta k_{t+1} = 0$ and $A_t = 1$ for each time period. Hence we can drop the time subscript in the steady state to obtain

$$\beta[\alpha k^{\alpha-1} + 1 - \delta] = 1.$$

As shown in Chapters 2 and 3, this implies that in equilibrium

$$\begin{aligned} k &\simeq \left(\frac{\delta+\theta}{\alpha}\right)^{\frac{-1}{1-\alpha}} \\ c &= k^{\alpha} - \delta k. \end{aligned}$$

We also note that the implied real interest rate is

$$r_t = ak_t^{\alpha-1} - \delta,$$

and so in equilibrium $r = \theta$, which is the required condition for a steady-state solution.

(iii) The optimal short-run solution to the model is a non-linear system of equations in two endogenous variables c_t and k_t. A log-linear approximation to this system is obtained using the result that the function $f(x_t)$ can be approximated about x_t^* as a linear relation in $\ln x_t$ by taking a first-order Taylor series approximation of

$$f(x_t) = f(e^{\ln x_t})$$

so that

$$\ln f(x_t) \simeq \ln f(x_t^*) + \frac{f'(x_t^*)x_t^*}{f(x_t^*)}[\ln x_t - \ln x_t^*].$$

This approximation can be generalised to $f(x_{1t},..., x_{nt})$ as

$$\ln f(x_{1t}, ..., x_{nt}) \simeq \ln f(x_{1t}^*, ..., x_{nt}^*) + \Sigma_{i=1}^n \frac{f_i'(x_{1t}^*, ..., x_{nt}^*)x_{it}^*}{f(x_{1t}^*, ..., x_{nt}^*)}[\ln x_{it} - \ln x_{it}^*].$$

Omitting the intercept, invoking certainty equivalence, noting that $E_t \ln A_{t+1} = \rho \ln A_t$ and in equilibrium $\ln A = 0$, the log-linear approximation to the Euler equation can be shown to be

$$E_t \Delta \ln c_{t+1} \simeq -\frac{(\delta+\theta)(1-\alpha)}{\sigma} E_t \ln k_{t+1} + \frac{(\delta+\theta)\rho}{\sigma} \ln A_t.$$

Omitting the intercept once again, the log-linearized resource constraint is

$$\ln k_{t+1} \simeq -\frac{\theta+(1-\alpha)\delta}{\alpha} \ln c_t + (1+\theta)\ln k_t + \frac{(\delta+\theta)\rho}{\alpha} \ln A_t$$

These two equations form a system linear in $\ln c_t$ and $\ln k_t$, from which the solution for each variable may be obtained.

(b) (i) In the above solution output, consumption and capital are stationary I(0) variables but in practice they are non-stationary I(1) variables. This alone implies that the model is misspecified.

(ii) A simple way to re-specify the model so that these variables are I(1) is to assume that technical progress is non-stationary, instead of stationary as at present. This can be achieved by setting $\rho = 1$. This assumption is sufficient to account for the non-stationarity of all real macroeconomic variables in the economy. It is, in effect, a common non-stationary stochastic shock. It does not, of course, explain why technical progress has this property. An extra source of non-stationarity in nominal macro variables is often assumed to come from the growth of the money supply. In countries where there is sustained population growth, for a time, this may generate an additional source of non-stationarity to all real macro variables.

(c) (i) In the model above output, consumption and capital have a single source of randomness, namely, technical progress. This implies that they are perfectly correlated. In practice, we find that they are highly, but not perfectly, correlated and are cointegrated. This implies that each variable must have a separate, or idiosyncratic, but stationary, source of randomness, but they have just the one non-stationary common stochastic shock.

(ii) Identifying what these additional sources of randomness are is a major feature in DSGE model building, especially where the aim is to estimate the model. A simple, and obvious, solution is to assume that all variables are measured with error, but the orders of magnitude of the required measurement errors make this somewhat implausible. Examples of other sources of randomnesss include preference shocks, additional independent stationary shocks to the production function, shocks due to obsolesence, independent shocks to the labour supply, price and wage setting shocks perhaps reflecting variable markups, shocks to government expenditures and revenues, domestic monetary shocks, foreign demand, price and interest rate shocks and exchange rate shocks.

14.2 After (log-) linearization all DSGE models can be written in the form

$$B_0 x_t = B_1 E_t x_{t+1} + B_2 z_t.$$

If there are lags in the model, then the equation will be in companion form and x_t and z_t will be long (state) vectors. And if B_0 is invertible then the DSGE model can also be written as

$$x_t = A_1 E_t x_{t+1} + A_2 z_t,$$

where $A_1 = B_0^{-1} B_1$ and $A_2 = B_0^{-1} B_2$.

(a) Show that the model in Exercise 14.1 can be written in this way.

(b) Hence show that the solution can be written as a vector autoregressive-moving average (VARMA) model.

(c) Hence comment on the effect of a technology shock.

Solution

(a) The log-linear approximation to the short-run solution of Exercise 14.1 can be written in matrix form as the system

$$\begin{bmatrix} 1+\theta & -\frac{\theta+(1-\alpha)\delta}{\alpha} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ln k_t \\ \ln c_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{(\delta+\theta)(1-\alpha)}{\sigma} & 1 \end{bmatrix} E_t \begin{bmatrix} \ln k_{t+1} \\ \ln c_{t+1} \end{bmatrix} - \begin{bmatrix} \frac{(\delta+\theta)\rho}{\alpha} \\ \frac{(\delta+\theta)\rho}{\sigma} \end{bmatrix} \ln A_t.$$

This may be denoted by the matrix equation

$$B_0 x_t = B_1 E_t x_{t+1} + B_2 z_t,$$

where $X_t = (\ln k_t, \ln c_t)'$ and $Z_t = \ln A_t$. As B_0 is invertible, the system can also be written as

$$x_t = AE_t x_{t+1} + Cz_t,$$

where $A = B_0^{-1}B_1$ and $C = B_0^{-1}B_2$.

(b) To show that the system can be written as a VAR we must first examine its dynamic properties. Introducing the lag operator and recalling that $E_t x_{t+1} = L^{-1}x_t$, the system can be written as

$$(I - AL^{-1})x_t = Cz_t.$$

The dynamic solution of this equation depends on the roots of the determinental equation

$$|A| - (trA)L + L^2 = 0.$$

There are two roots. Setting $L = 1$ gives three cases:

(i) $|A| - (trA) + 1 > 0$ implies both roots are either stable or unstable,

(ii) $|A| - (trA) + 1 < 0$ implies a saddlepath solution (one root is stable and the other is unstable),

(iii) $|A| - (trA) + 1 = 0$ then at least one root is 1.

Assuming that $\sigma \geq \alpha + 2$ - a sufficient but not necessary condition - it can be shown that

$$|A| - (trA) + 1 = -\frac{[\theta + (1-\alpha)\delta]\frac{(\delta+\theta)(1-\alpha)}{\sigma}}{\alpha(1+\theta)} < 0.$$

Thus, the two roots satisfy $\eta_1 > 1$ and $\eta_2 < 1$, and so the short-run dynamics about the steady-state growth path follow a saddlepath. The system can be written as

$$\eta_1(1 - \frac{1}{\eta_1}L)(1 - \eta_2 L^{-1})x_t = adj(A - L)Cz_t.$$

Hence,

$$x_t = \frac{1}{\eta_1}x_{t-1} + \frac{1}{\eta_1}(1 - \eta_2 L^{-1})^{-1} adj(A - L)Cz_t.$$

Recalling that $z_t = \ln A_t = \rho \ln A_{t-1} + e_t$ and $E_t \ln A_{t+s} = \rho^s \ln A_t$, we can re-write this as

$$x_t = \frac{1}{\eta_1} x_{t-1} + G_0 \ln A_t + G_1 \ln A_{t-1},$$

or

$$x_t = (\frac{1}{\eta_1} + \rho) x_{t-1} - \frac{\rho}{\eta_1} x_{t-2} + G_0 e_t + G_1 e_{t-1},$$

which is a VARMA(2,1) in x_t.

(c) It follows that a technology shock e_t will cause a disturbance to equilibrium and that the system will return to equilibrium at a speed dependent on η_1 and ρ. The closer is ρ to unity, the slower the return to equilibrium. If $\rho = 1$ then the solution would take the form

$$\Delta x_t = \frac{1}{\eta_1} \Delta x_{t-1} + G_0 e_t + G_1 e_{t-1}.$$

The shock would then have a permanent effect on x_t and solution would be a VARIMA(1,1,1).

14.3. Consider the real business cycle model defined in terms of the same variables as in Exercise 14.1 with the addition of employment, n_t:

$$\begin{aligned}
\mathcal{U}_t &= E_t \sum_{s=0}^{\infty} \beta^s [\frac{c_{t+s}{}^{1-\sigma}}{1-\sigma} - \gamma \frac{n_{t+s}{}^{1-\phi}}{1-\phi}] \\
y_t &= c_t + i_t \\
y_t &= A_t k_t^{\alpha} n_t^{1-\alpha} \\
\Delta k_{t+1} &= i_t - \delta k_t \\
\ln A_t &= \rho \ln A_{t-1} + e_t,
\end{aligned}$$

where $e_t \sim i.i.d(0, \omega^2)$.

(a) Derive the optimal solution

(b) Hence find the steady-state solution.

(c) Log-linearize the solution about its steady state to obtain the short-run solution.

(d) What is the implied dynamic behavior of the real wage and the real interest rate?

Solution

(a) The economy's resource constraint is

$$A_t k_t^{\alpha} n_t^{1-\alpha} = c_t + k_{t+1} - (1-\delta)k_t$$

The problem is to maximize the value function

$$\begin{aligned} V_t &= U(c_t) + \beta E_t(V_{t+1}) \\ &= \frac{c_{t+s}{}^{1-\sigma}}{1-\sigma} - \gamma \frac{n_{t+s}{}^{1-\phi}}{1-\phi} + \beta E_t(V_{t+1}) \end{aligned}$$

subject to the resource constraint. Noting the results in Exercise 14.1, the first-order conditions are

$$\frac{\partial V_t}{\partial c_t} = c_t^{-\sigma} - \beta E_t[c_{t+1}^{-\sigma} \cdot \alpha A_{t+1} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + 1 - \delta] = 0$$

and

$$\frac{\partial V_t}{\partial n_t} = n_t^{-\phi} + \beta E_t[\frac{\partial V_{t+1}}{\partial n_{t+1}} \cdot \frac{\partial n_{t+1}}{\partial n_t}] = 0$$

with

$$\frac{\partial V_{t+1}}{\partial n_{t+1}} = n_{t+1}^{-\phi}.$$

From the resource constraint

$$\begin{aligned} \frac{\partial n_{t+1}}{\partial n_t} &= \frac{\frac{\partial n_{t+1}}{\partial k_{t+1}}}{\frac{\partial n_t}{\partial k_{t+1}}} \\ &= -\frac{\frac{\alpha A_{t+1} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + 1 - \delta}{(1-\alpha) A_{t+1} k_{t+1}^{\alpha} n_{t+1}^{-\alpha}}}{[(1-\alpha) A_t k_t^{\alpha} n_t^{-\alpha}]^{-1}} \\ &= -[\alpha A_{t+1} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + 1 - \delta] \frac{A_t}{A_{t+1}} \left(\frac{k_t}{k_{t+1}}\right)^{\alpha} \left(\frac{n_{t+1}}{n_t}\right)^{\alpha}, \end{aligned}$$

giving

$$\frac{\partial V_t}{\partial n_t} = n_t^{-\phi} - \beta E_t\{n_{t+1}^{-\phi}[\alpha A_{t+1} k_{t+1}^{\alpha} n_{t+1}^{-\alpha} + 1 - \delta] \frac{A_t}{A_{t+1}} \left(\frac{k_t}{k_{t+1}}\right)^{\alpha} \left(\frac{n_{t+1}}{n_t}\right)^{\alpha}\} = 0.$$

Hence the solution is

$$
\begin{aligned}
E_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}(\alpha A_{t+1}k_{t+1}^{\alpha-1}n_{t+1}^{1-\alpha}+1-\delta)\right] &= 1 \\
E_t\left[\beta\frac{A_t}{A_{t+1}}\left(\frac{k_t}{k_{t+1}}\right)^{\alpha}\left(\frac{n_t}{n_{t+1}}\right)^{\phi-\alpha}(\alpha A_{t+1}k_{t+1}^{\alpha-1}n_{t+1}^{1-\alpha}+1-\delta)\right] &= 1
\end{aligned}
$$

plus the economy's resource constraint.

Note that this result appears to be different from that in Chapter 2 as here we are taking account of the stochastic structure of the problem and therefore using stochastic dynamic programming. The second equation is an Euler equation for employment. If combined with the Euler equation for consumption it would give a similar result to that in Chapter 2.

(b) The steady-state solution - assuming it exists - satisfies $\Delta c_{t+1} = \Delta k_{t+1} = \Delta n_{t+1} = 0$ and $A_t = 1$ for each time period. Hence we can drop the time subscript in the steady state to obtain from both equations above

$$\beta[\alpha k^{\alpha-1}n^{1-\alpha}+1-\delta] = 1.$$

In equilibrium therefore

$$
\begin{aligned}
\frac{k}{n} &\simeq \left(\frac{\delta+\theta}{\alpha}\right)^{\frac{-1}{1-\alpha}} \\
\frac{c}{n} &= \left(\frac{k}{n}\right)^{\alpha} - \delta\frac{k}{n}.
\end{aligned}
$$

This shows that we can only determine the long-run values of $\frac{c}{n}$ and $\frac{k}{n}$.

(c) Omitting the intercept, invoking certainty equivalence, noting that $E_t \ln A_{t+1} = \rho \ln A_t$ and in equilibrium $\ln A = 0$, the log-linear approximation to the above solution is

$$
\begin{aligned}
E_t\Delta\ln c_{t+1} &\simeq -\frac{(\delta+\theta)(1-\alpha)}{\sigma}(E_t\ln k_{t+1} - E_t\ln n_{t+1}) + \frac{(\delta+\theta)\rho}{\sigma}\ln A_t \\
E_t\Delta\ln n_{t+1} &\simeq -\frac{\alpha}{\phi-\alpha}E_t\Delta\ln k_{t+1} + \frac{(\delta+\theta)(1-\alpha)}{\phi-\alpha}(E_t\ln k_{t+1} - E_t\ln n_{t+1}) - \frac{(\delta+\theta)\rho}{\phi-\alpha}\ln A_t
\end{aligned}
$$

together with the log-linearized resource constraint

$$\ln k_{t+1} \simeq -\frac{\theta+(1-\alpha)\delta}{\alpha}\ln c_t + (1+\theta)\ln k_t + \frac{(\delta+\theta)(1-\alpha)}{\alpha}\ln n_t + \frac{(\delta+\theta)\rho}{\alpha}\ln A_t.$$

These three equations form a system linear in $\ln c_t$, $\ln k_t$ and $\ln n_t$ from which the short-run solution for each variable may be obtained. The resulting system can now be represented as a VARMA model in the three variables.

(d) From Chapter 2, the implied real interest rate and real wage are

$$\begin{aligned} r_t &= \frac{\partial y_t}{\partial k_t} - \delta \\ &= \alpha A_t \left(\frac{k_t}{n_t}\right)^{-(1-\alpha)} - \delta \\ w_t &= \frac{y_t - (r_t + \delta)k_t}{n_t} \\ &= (1-\alpha)A_t \left(\frac{k_t}{n_t}\right)^{\alpha}. \end{aligned}$$

Their dynamic behavior can therefore be derived directly from that of $\frac{k_t}{n_t}$ of $\ln k_t - \ln n_t$.

14.4 For a log-linearized version of the model of Exercise 14.1 write a Dynare program to compute the effect of an unanticipated temporary technology shock on the logarithms of output, consumption and capital and the implied real interest rate assuming that $\alpha = 0.33$, $\delta = 0.1$, $\sigma = 4$, $\theta = 0.05$, $\rho = 0.5$ and the variance of the technology shock e_t is zero.

Notes:

(i) Dynare runs in both Matlab and Gauss and is freely downloadable from http://www.dynare.org/

(ii) Dynare uses a different dating convention. It dates non-jump variables like the capital stock at the end and not the start of the period, i.e. as $t-1$ and not t.

Solution

The different dating convention of Dynare implies the solution of Exercise 14.1 must be re-dated as

$$\begin{aligned} A_t k_{t-1}^{\alpha} &= c_t + k_t - (1-\delta)k_{t-1} \\ E_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}(\alpha A_{t+1}k_t^{\alpha-1} + 1 - \delta)\right] &= 1 \end{aligned}$$

The log-linear approximation to this model is

$$
\begin{aligned}
E_t \Delta \ln c_{t+1} &\simeq -\frac{(\delta+\theta)(1-\alpha)}{\sigma} E_t \ln k_t + \frac{(\delta+\theta)}{\sigma} \ln A_{t+1} \\
\ln k_t &\simeq -\frac{\theta+(1-\alpha)\delta}{\alpha} \ln c_t + (1+\theta) \ln k_{t-1} + \frac{(\delta+\theta)}{\alpha} \ln A_t \\
\ln A_t &= \rho \ln A_{t-1} + e_t
\end{aligned}
$$

The real interest rate is

$$r_t = \alpha A_t k_{t-1}^{-(1-\alpha)} - \delta.$$

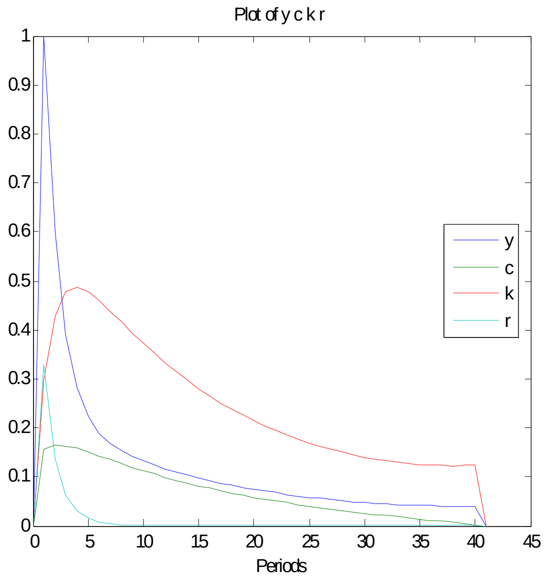
We define the variables as log deviations from their steady state, hence their initial values are set to zero. The Dynare program is

```
% Ex14.4
close all;
%————————————
% variables
%————————————
var y c k a r;
varexo e;
%————————————
% parameters
%————————————
parameters delta theta alpha sigma rho;
alpha=0.33;
theta=0.05;
sigma= 4;
delta=0.1;
rho=0.5;
```

```
%————————
% model
%————————
model;
c=c(+1)+((delta+theta)*(1-alpha)/sigma)*k-((delta+theta)/sigma)*a(+1);
k=-((theta+delta*(1-alpha))/alpha)*c+(1+theta)*k(-1)+((delta+theta)/alpha)*a;
y=alpha*k(-1)+a;
a=rho*a(-1)+e;
r=alpha*a*exp((alpha-1)*k(-1));
end;
%————————————-
% initial values of variables
%————————————-
initval;
y=0;
k=0;
c=0;
e=0;
i=0;
end;
steady;
%————————
% values of lagged variables
%————————
histval;
k(0)=0;
```

```
a(0)=0;
end;
%————————
% shocks
%————————
shocks;
var e;
periods 1;
values 1;
end;
%————————
% computation
%————————
simul (periods=40);
rplot y c k r;
```

The impulse response functions, which are cut off after 40 periods and forced to converge to their steady-state, are



14.5 For the model of Exercise 14.1 write a Dynare program to compute the effect of a temporary technology shock assuming that $\alpha = 0.33$, $\delta = 0.1$, $\sigma = 4$, $\theta = 0.05$, $\rho = 0.5$ and the variance of the technology shock e_t is unity. Plot the impulse response functions for output, consumption, capital and the real interest rate.

Solution

The different dating convention of Dynare implies the solution of Exercise 14.1 must be re-dated as

$$\begin{aligned} A_t k_{t-1}^{\alpha} &= c_t + k_t - (1-\delta)k_{t-1} \\ E_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}(\alpha A_{t+1}k_t^{\alpha-1} + 1 - \delta)\right] &= 1 \end{aligned}$$

We must then add the equation

$$\ln A_t = \rho \ln A_{t-1} + e_t.$$

Finally, we recall that the real interest rate is

$$r_t = \alpha A_t k_{t-1}^{-(1-\alpha)} - \delta.$$

The initial and steady-state values are calculated from the parameter values. Thus, $A = 1$, $k = \left(\frac{\delta+\theta}{\alpha}\right)^{\frac{1}{\alpha-1}}$, $y = k^a$ and $c = y - \delta k$. The Dynare program is then

```
% Ex14.5

close all;

%——————————

% variables

%——————————

var y c k a r;

varexo e;

%——————————

% parameters

%——————————

parameters delta theta alpha sigma rho;

alpha=0.33;

theta=0.05;

sigma= 4;

delta=0.1;

rho=0.5;

%——————————

% model

%——————————

model;

c=c(+1)*(((alpha*a(+1)*(k^(alpha-1))+1-delta)/(1+theta))^(-1/sigma));

y=a*(k(-1)^alpha);

k=y-c+(1-delta)*k(-1);
```

```
ln(a)=rho^ln(a(-1))+e;
r=alpha*a*(k(-1)^(alpha-1));
end;
%——————————————-
% initial values of variables
%——————————————-
initval;
k=3.244;
c=1.15;
a=1;
e=0;
end;
steady;
%————————————
% values of lagged variables
%————————————
histval;
k(0)=3.244;
a(0)=1;
end;
%————————————
% shocks
%————————————
shocks;
var e;
periods 1;
```

values 1;

end;

%——————————

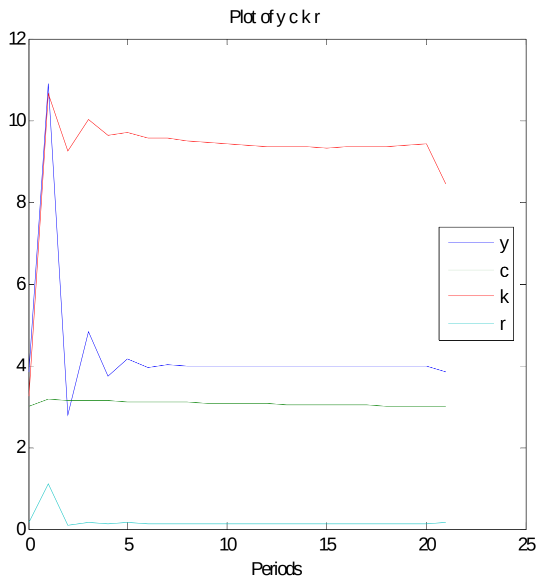
% computation

%——————————

simul (periods=20);

rplot y c k r;

The impulse response functions are



14.6. For the model of Exercise 14.5 write a Dynare program for a stochastic simulation which calculates the means, variances, cross correlations and autocorrelations.

Solution

The Dynare program is

% Ex14.6

```
close all;

%——————————

% variables

%——————————

var y c k a r;

varexo e;

%——————————

% parameters

%——————————

parameters delta theta alpha sigma rho;

alpha=0.33;

theta=0.05;

sigma= 4;

delta=0.1;

rho=0.5;

%——————————

% model

%——————————

model;

c=c(+1)*(((alpha*a(+1)*(k^(alpha-1))+1-delta)/(1+theta))^(-1/sigma));

y=a*(k(-1)^alpha);

k=y-c+(1-delta)*k(-1);

ln(a)=rho^ln(a(-1))+e;

r=alpha*a*(k(-1)^(alpha-1));

end;

%————————————-
```

```
% initial values of variables
%————————————-
initval;
k=3.244;
c=1.15;
a=1;
e=0;
end;
steady;
%—————————
% values of lagged variables
%—————————
histval;
k(0)=3.244;
a(0)=1;
end;
%—————————
% shocks
%—————————
shocks;
var e = 1;
end;
steady;
stoch_simul(order = 1);
```

The output is

THEORETICAL MOMENTS

VARIABLE	MEAN	STD. DEV.	VARIANCE
y	3.8395	4.2289	17.8835
c	2.9948	0.8383	0.7028
k	8.4469	6.9681	48.5540
a	1.8987	2.1196	4.4926
r	0.1500	0.1995	0.0398

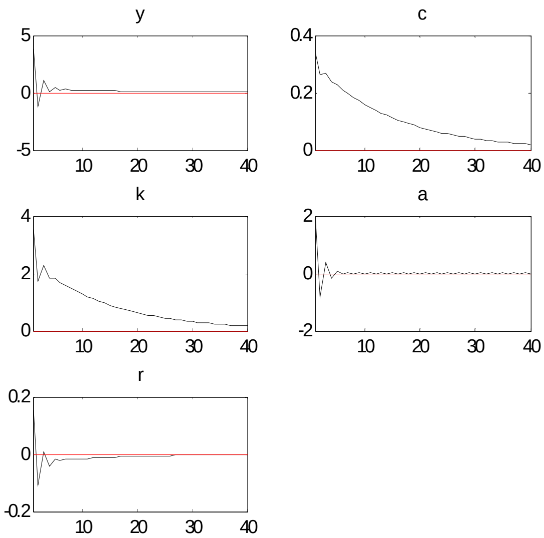
MATRIX OF CORRELATIONS

Variables	y	c	k	a	r
y	1.0000	0.5035	0.6080	0.9700	0.7856
c	0.5035	1.0000	0.9921	0.2782	-0.1391
k	0.6080	0.9921	1.0000	0.3966	-0.0136
a	0.9700	0.2782	0.3966	1.0000	0.9125
r	0.7856	-0.1391	-0.0136	0.9125	1.0000

COEFFICIENTS OF AUTOCORRELATION

Order	1	2	3	4	5
y	-0.2866	0.2746	0.0152	0.1212	0.0655
c	0.9124	0.8599	0.7980	0.7460	0.6949
k	0.8337	0.8215	0.7466	0.7050	0.6536
a	-0.4444	0.1975	-0.0878	0.0390	-0.0173
r	-0.3347	0.2511	-0.0162	0.0962	0.0403

The responses to the shock are



14.7 (a) Consider the New Keynesian model

$$\begin{aligned}
\pi_t &= \pi^* + \alpha(E_t\pi_{t+1} - \pi^*) + \beta(\pi_{t-1} - \pi^*) + \delta x_t + e_{\pi t} \\
x_t &= E_t x_{t+1} - \gamma(R_t - E_t\pi_{t+1} - \theta) + e_{xt} \\
R_t &= \theta + \pi^* + \mu(\pi_t - \pi^*) + \upsilon x_t,
\end{aligned}$$

where π_t is inflation, π^* is target inflation, x_t is the output gap, R_t is the nominal interest rate, $e_{\pi t}$ and e_{xt} are independent, shocks.ean iid processes and $\phi = (1-\beta)\pi^*$. Write a Dynare program to compute the effect of a supply shock in period t such that $e_{\pi t} = -e_{xt} = 5$. Assume that $\pi^* = 2$, $\alpha = 0.6$, $\alpha + \beta = 1$, $\delta = 1$, $\gamma = 5$, $\theta = 3$, $\mu = 1.5$ and $\nu = 1$.

(b) Compare the monetary policy response to the increase in inflation compared with that of a strict inflation targeter when $\nu = 0$.

Solution

(a) The Dynare program is

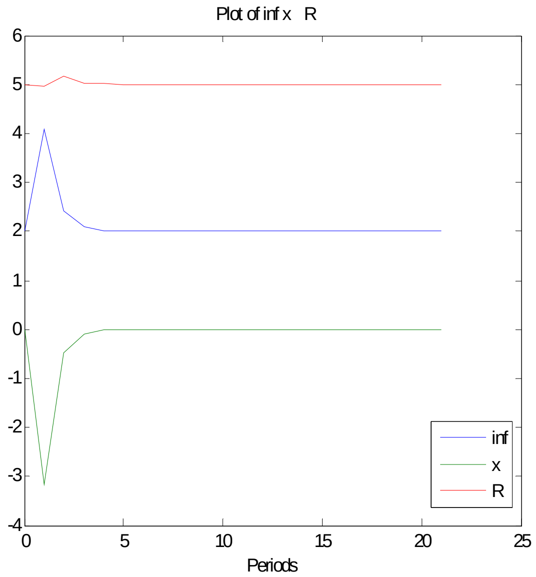
% Ex14.7

```
% New Keynesian model with a Taylor rule

% The effect of a supply shock

close all;

%——————————-

% variables

%——————————

var inf x R;

varexo e;

%——————————-

% parameters

%——————————

parameters alpha beta delta theta mu nu;

alpha=0.6;

beta=1-alpha;

delta=1;

theta=5;

mu=1.5;

nu=1;

%——————————-

% model

%——————————

model(linear);

inf = 2 + alpha*(inf(+1)-2) + beta*(inf(-1)-2) + delta*x + e;

x = x(+1) - theta*(R-inf(+1)-3)-e;

R = 5 + mu*(inf-2) + nu*x;

end;
```

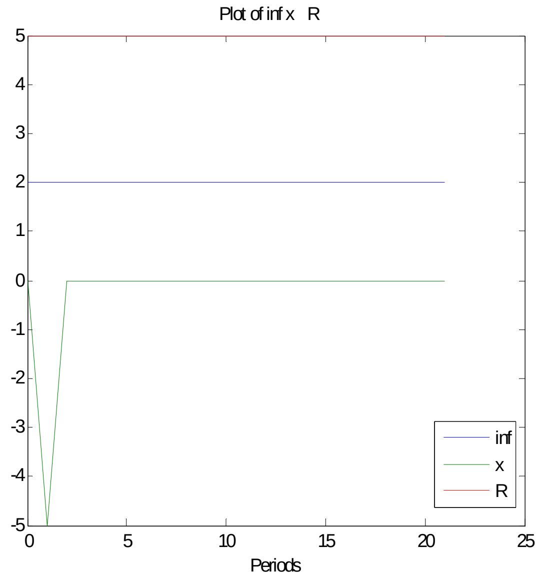
```
%--------------------------------------------
% initial values
%--------------------------------------------
initval;
inf=2;
x=0;
R=5;
e=0;
end;
steady;
%--------------------------------------------
% values of lagged variables
%--------------------------------------------
histval;
inf(0)=2;
end;
%--------------------------------------------
% shocks
%--------------------------------------------
shocks;
var e; periods 1; values 5;
end;
%--------------------------------------------
% computation
%--------------------------------------------
simul(periods=20);
```

rplot inf x R;

(b) The impulse response functions for flexible and strict inflation targeting are



Flexible inflation targeting



Strict inflation targeting

With flexible inflation targeting monetary policy hardly reacts to the increase in inflation. But with strict inflation targeting monetary policy reacts strongly to offset the inflation shock leaving inflation at its target value. As a result output falls more with strict inflation targeting than with flexible inflation targeting.